UNIVERSITY OF ALBERTA

# On Some Tests For The Change Point Problem

BY

ABD-ELNASER S. ABD-RABOU ©

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## Abstract

This thesis has four main chapters in which we discuss four different change point problems. In Chapter 2, we developed non-parametric weighted Least Squares tests for a possible change in the slope of a simple regression model. We also proposed Least Squares tests for the epidemic -type alternative and for the at most two change points alternative in the slope of linear model. In Chapter 3, we used Bayesian approach to develop a test to detect an epidemic-type change in the parameters of the general linear models. Chapter 4, focuses on introducing simple rank tests to detect a distributional change in samples with random size. Finally, in Chapter 5 we obtained non-parametric tests for the multiple change points under ordered alternatives.

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TO MY FRIEND NISHECKO

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# Chapter 1

## Introduction

In many practical and experimental situations some statistical properties of an observed phenomenon may change abruptly at some unknown time point(s). The detection and characterization of such a change are problems of interest in many scientific fields. Examples can be found in speech signals recognition, epidemiology (incidence of a disease) and quality control studies. In statistical literature such problems are called "change point" problems. In the past three decades an extensive amount of research has been done in this area using different approaches to treat the problem in both parametric and nonparametric contexts. For review we refer to Shaban (1980), Basseville and Benveniste (1986) and Lombard (1989) as a classical treatment, to Broemeling and Tsurumi (1987) as Bayesian treatment and to Zacks (1983) for Bayesian and non-Bayesian survey.

To introduce the change point problem, let us consider the following simple case of "at most one change" (AMOC) in the distribution function (DF) of a sequence of random variables (rv's) . Given a time-ordered sequence of independent observations $\mathbf{X} = (X_1, \ldots, X_n)$ with corresponding DF's $F_1, \ldots, F_n$. In a conventional statistical analysis we generally assume that $F_1 = F_2 = \ldots = F_n$ which means that $\mathbf{X}$ is a random sample from a common DF, $F$ (say). However, we may for some reason(s) suspect that a change has taken place in the DF of the $X_i$'s at some unknown point $m$, $1 \leq m \leq n-1$. This problem can be written in terms of the following hypotheses;

$$H_o : F_1 = \ldots = F_n = F, \qquad (F \text{ is usually unknown})$$

against, (1.1)

$$H_1 : \exists \text{ unknown } \tau \in (0,1) \text{ and G} \neq \text{F such that}$$

$$F_1 = \ldots = F_{[n\tau]} = F \neq F_{[n\tau]+1} = \ldots = F_n = G,$$

where $[y] :=$ the integer part of $y$. In practice, more complicated situations than the one described above can arise. For example, there are often grounds for suspecting that more than one change point may be present. In this case we are testing $H_o$ against the multiple change in distribution alternative hypothesis;

$$\begin{aligned} H_2 \quad : \quad & \exists \text{ unknown } 0 < \tau_1 < \tau_2 < \ldots < \tau_r < 1, \text{ r} < \text{n such that} \\ & F_i = G_1, \qquad 1 \leq i \leq [n\tau_1], \\ & F_i = G_2, \qquad [n\tau_1] < i \leq [n\tau_2], \\ & \vdots \\ & F_i = G_r, \qquad [n\tau_r] < i \leq n. \end{aligned} \tag{1.2}$$

The "at most two change points" (AMTC), i.e. $r = 2$ is the most commonly treated case in the literature since the generalization for $r > 2$ often follows from $r = 2$. A well known, special case of the AMTC problem is the epidemic (square) change point problem. The epidemic type alternative of change in DF's takes the form:

$$H_3 \; : \; \exists \text{ unknown } 0 < \tau_1 < \tau_2 < 1 \text{ and } \mathrm{G}_1 \neq \mathrm{G}_2 \text{ such that}$$

$$F_i = G_1, \qquad i \leq [n\tau_1] \text{ or i} > [\mathrm{n}\tau_2],$$

and,

$$F_i = G_2, \qquad [n\tau_1] < i \leq [n\tau_2]. \tag{1.3}$$

After this brief description of the change point problem, (if a change occurs in the DF), it is appropriate to describe the problems considered in this dissertation. This thesis has four parts (problems). The first two problems deal with making inference about changes which occur in the parameters of linear models and the last two, deal with making inference about changes in a sequence of DF's.

**Problem 1**

In this part we discuss the problem of detecting a possible change in the slope of a simple regression model. A sequence of independent rv's $Y_1, \ldots, Y_n$, generated over successive time points $x_1 < x_2 < \ldots < x_n$, is given. Under the null hypothesis, the $Y$'s follow the model

$$Y_i = \alpha + \beta x_i + \epsilon_i \qquad ,i = 1, \ldots, n, \tag{1.4}$$

where $\alpha$, the intercept and $\beta$, the slope are assumed unknown. The errors $\epsilon_1, \ldots, \epsilon_n$ are iid with mean zero and finite variance $\sigma^2$. Here we are interested in testing whether or not there is a change in the regression slope $\beta$ at an unknown time point $m$ ,$m = 1, \ldots, n-1$. This problem is equivalent to testing if the regression model has changed to :

$$Y_i = \alpha + \begin{cases} \beta x_i + \epsilon_i & \text{for } i = 1, \ldots, m \\ \gamma x_i + \epsilon_i & \text{for } i = m+1, \ldots, n \,, \end{cases} \tag{1.5}$$

where $\beta \neq \gamma$ and $m$ are unknown. The above testing problem can also be represented by the hypotheses

$$H_o : \beta = \gamma \qquad \text{vs} \qquad H_1 : \beta \neq \gamma. \tag{1.6}$$

Classical (non-Bayesian) tests for shifts in regression models can be divided into two main streams. The first group of tests are those which assume that the regression errors are normally distributed. Most of these tests are based on the likelihood ratio (LR) statistic. On the other hand nonparametric tests which assume no specific distribution are based either on the least squares (LS) residuals or on ranks. First, we mention here some of the related work done under the normality assumption. Quandt (1958, 1960) used the maximum likelihood (ML) technique to estimate the change point and to test for a change in both parameters of a simple linear model. Hinkley (1969, 1971) discussed the problem of estimating and making inference about the intersection of two simple regression lines through the ML technique. Farley and Hinich (1970) derived the LR test for a shift in the slope of a simple regression model when the size of the shift is relatively small with respect to the error variance. Brown *et al.* (1975) in a famous paper introduced a recursive residual procedure to test for a change in multiple regression parameters. Maronna and Yohai (1978) derived the MLR test for a change in the intercept term and obtained through simulation empirical quantiles for their test statistic. Hawkins (1980) pointed out that the LR test for a change in the slope of a simple linear model does not converge in the case of a discontinuous change point model. Worsley (1983) gave good approximate upper bounds for the null distribution of the LR test in the multiple regression model. Kim and Siegmund (1989) introduced LR tests for change in the parameters of a simple regression model and obtained approximations for the asymptotic distributions of their test statistics which worked well in large samples. Horváth (1995) obtained the asymptotic distribution of the LR test in multiple regression model and proved the consistency of the proposed test. Now we mention some of the related nonparametric work. Sen (1980) introduced tests for a possible change in the slope of the simple regression model based on the LS estimators and derived their asymptotic distributions under the null hypothesis as well as under contiguous alternatives. Hušková and Sen (1984) proposed rank tests for a change in the multiple regression parameters. Gombay and Horváth (1994) proved limit theorems for tests based on LS residuals in multiple regression models. Finally, Miao (1988) proposed a very simple test based on the difference of cumulative sums of the dependent variable $\mathbf{Y}$, to test for a change in the slope of the simple regression model and derived its limiting distribution under both normal and non-normal errors.

**Problem 2**

Here we are interested in making inference about a possible epidemic-type change in the parameters of the general linear model (GLM). Let $Y_1, \ldots, Y_n$ be a sequence of independent observations taken in a successive manner (e.g. over time) and obey the regression model,

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon, \tag{1.7}$$

where $\mathbf{Y} = (Y_1, \ldots, Y_n)', \varepsilon = (\epsilon_1, \ldots, \epsilon_n)' \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$, $\beta = (\beta_0, \ldots, \beta_m)$ is the parameter vector and $\mathbf{X}$ is the design matrix. The epidemic (square) type change in $\beta$ can be represented by the hypotheses

$H_\circ$ : The $Y$'s follow the regression model in (1.7),

against

$H_1$ : For some $\delta$ , represents the amount of change , the model is

$$\mathbf{Y} = \mathbf{X}\beta + \delta \sum_{j=0}^{m} \mathbf{X}_{k,j}^{l} + \varepsilon, \tag{1.8}$$

where $0 \leq k < l \leq n-1$ are unknown and

$$\mathbf{X}_{k,j}^{l} = (0, \ldots, 0, x_{k+1,j}, \ldots, x_{l,j}, 0, \ldots, 0)'. \tag{1.9}$$

The above hypotheses are equivalent to

$$H_\circ : \delta = 0 \qquad \text{vs} \qquad H_1 : \delta \neq 0. \tag{1.10}$$

These type of alternatives (i.e. the epidemic-type) were first introduced by Levin and Kline (1985) in their study of spontaneous abortions. Lombard (1987), Aly and Bouzar (1992) and Gombay (1994) proposed rank and LR tests for the square-type change that may occur in a sequence of DF's. Since our intention is to solve the above problem using a Bayesian technique we will focus our attention here only on the related Bayesian work done in this area. In Bayesian analysis there are two main approaches used to make inference about change in linear models. Assume that a parameter change has taken place at some time point. The first approach utilizes the marginal posterior distributions to estimate the change point in linear models. Holbert (1973) was the first to complete a through analysis for making inferences about the parameters of a changing linear model. He derived the posterior distribution, jointly, for all the parameters and marginally for the switch point and intersection point, using vague-type priors. With Uniform prior assigned to the shift point, Holbert and Broemeling (1977) derived the posterior, jointly for all the parameters and marginally for the shift point in simple regression. Choy and Broemeling (1980), derived the posterior probability mass function for the change point and used it to determine the most probable value of the shift point in general linear models. Wang and Lee (1993), employed a non-informative prior to the change point and used its corresponding posterior distribution to determine the most probable change point position when the change occurs in the intercept of a simple linear regression. For more details about this approach we refer to Broemeling and Tsurumi (1987).

The second approach constructs Bayesian change point test statistics using the so called Bayesian likelihood ratio (BLR) method. This method was first introduced by Chernoff and Zacks (1964), to derive test statistics for the one-sided detection of parameter changes at unknown time in the mean of a sequence of independent normal variables. Their methodology, is based on assuming suitable prior distributions for the nuisance parameters under the null as well as the alternative hypotheses. Then the unconditional likelihood functions are obtained by integrating out the nuisance parameters. Finally from the LR, the BLR-type test statistic is obtained. This technique was used by Gardner (1969) to detect two-sided parameter change in a sequence of normal variables. MacNeill (1974) derived two-sided BLR test statistics to test for a change in the parameters of a sequence of random variables from the exponential family. Jandhyala and MacNeill (1987), derived BLR test statistics to detect one-sided and two-sided changes in the parameters of the GLM. Their test statistics for the one-sided and two-sided changes are linear and quadratic functions in the regression residuals, respectively. They also pointed out that the asymptotic theory, even of the two-sided tests, is generally complicated and only tractable to some extent. In (1989) they were able to determine the asymptotic distribution quantiles for the two-sided tests in the case of harmonic polynomial regression. Jandhyala and MacNeill (1991) computed the asymptotic quantiles in the case of first order polynomial regression and applied them in a numerical power study. Jandhyala and Minogue (1993) introduced a numerical procedure to solve the integral equations involved in computing the asymptotic quantiles of the two-sided BLR-type tests in polynomial regressions.

### Problem 3

Inferences drawn from samples with random size are important in many fields such as, biology, insurance and telephone engineering. Our objective in this part is to study simple rank tests for the two-sample problem when the sample sizes are random. We will also develop tests for the change point problem, based on samples with random size. These two problems, are described in the following parts.

### The two-sample problem when the sample sizes are random

Let $X_1, X_2, \ldots,$ and $Y_1, Y_2, \ldots$ be two independent sequences of independent random variables. The random variables $X$ and $Y$ have distribution functions $F$ and $G$, respectively. Let $\{M_m, m \geq 1\}$ and $\{N_n, n \geq 1\}$ be two independent sequences of nonnegative integer-valued random variables. based on the samples $\{X_i, 1 \leq i \leq M_m\}$ and $\{Y_j, 1 \leq j \leq N_n\}$ we wish to test the hypotheses;

$$H_\circ : F = G \qquad \text{vs} \qquad H_1 : F \neq G. \tag{1.11}$$

For this problem we extend the results of Aly *et al.* (1987) to the case when the sample sizes are random. We studied here rank tests of Chernoff-Savage type for the hypotheses in (1.11). The limiting distribution of the test statistic is obtained.

**The change point problem in a sample of random size**

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables and $\{N_n, n \geq 1\}$ be a sequence of nonnegative integer-valued random variables. Now, suppose that $X_1, X_2, \ldots, X_{N_n}$ is a sample of a random size $N_n, n \geq 1$. We study here the problem of testing the null hypothesis of no change against the at most one change point alternative,

$$H_1 \; : \; X_i \sim F \;\; , \;\; 1 \leq i \leq k \;\; \& \;\; X_i \sim G \;\; , i > k, \tag{1.12}$$

where $F \neq G$ and $k$ are unknown. As in the two-sample problem we propose a Chernoff-Savage type rank test statistic and derive its asymptotic distribution.

**Problem 4**

Change point analysis usually deals with what is called "general alternatives". That is, there exists one (or more) change point(s) for which the data sequence can be divided into two (or more) unequal groups with respect to some parameter. But in some situations the above unrestricted type of alternative is not suitable. For example (see Jonckheere (1954)), in testing the effect of stress on the task of manual dexterity, the data is taken from groups of subjects working under high, medium, low and minimal stress. Here the null hypothesis would be, stress has no effect on these subjects' performance and the suitable alternative is that, increasing stress will have increasing effect on their performance. These type of situations are called "ordered alternatives" problems. In this part of our research we are studying the change point problem under ordered alternatives which is described as follows. For simplicity, we assume that we are testing against the alternative of exactly two ordered changes in the location parameters. Let $X_1, \ldots, X_n$ be a sequence of independent random variables with unknown location parameters $\mu_1, \ldots, \mu_n$. The change point ordered alternative states that for unknown $0 < s < t < 1$, $\mu_1 = \ldots = \mu_{[ns]} < \mu_{[ns]+1} = \ldots = \mu_{[nt]} < \mu_{[nt]+1} = \ldots = \mu_n$. Jonckheere (1954) and Terpestra (1952) studied the k-sample version of this problem which corresponds to the case when the change points $[ns]$ and $[nt]$ are known. Based on the testing procedure of Jonckheere (1954) and Terpestra (1952) we develop a test statistic to detect the ordered type change in the location of a sequence of observations. We also generalize this situation to test for the ordered alternative changes in the distribution function rather than in the location parameter. The latter test is based on Puri (1965)'s multi-sample test.

This thesis is divided into four main Chapters. In each Chapter we investigate one of the above four problems. In Chapter 2, we generalize the LS approach of Sen (1980) by proving weighted approximations of certain LS change point processes. These results are then used to develop Cramér-von Mises, Anderson-Darling and Erdős-Darling type test statistics. The limiting distributions of the last two test statistics are derived. We also propose test statistics for testing against the two change points alternative and the epidemic-type alternative for change in the slope of a simple regression model. Asymptotic distributions and computing formulas for these tests are also provided. Finally in this chapter and through simulation, we obtain empirical critical values and Monte Carlo powers for Sen test (sup-type test), Cramér-von Mises, Anderson-Darling and Erdős-Darling type tests. We also estimate the critical values and powers of the epidemic-type tests and the AMTC test. In Chapter 3 we examine Problem 2 using a Bayesian approach. We derive here a Bayesian likelihood ratio (BLR) test for the epidemic-type change in the parameters of the general linear model. Under assumptions on the design matrix, we determine the asymptotic distribution of the test. As an application, three examples are given and the limiting distributions of the first two of them are theoretically determined. In the third example and through simulations, we calculate empirical critical values for the test, approximation for the distribution quantiles and compute Monte Carlo powers under different change positions and sizes. In Chapter 4, which has two parts, we consider Problem 3. In the first part we propose a rank test for the two sample problem when the sample sizes are random. We also derive the limiting distribution of the proposed test statistic. In part two we extend the idea of rank statistic to the change point problem when the sample size is random. We introduced a rank test statistic to detect a possible change in the DF's. The asymptotic distribution of this test is derived. In Chapter 5, we develop nonparametric tests for the ordered-type alternatives of problem 4. We first treat the case where the change occurs in the location parameter of a distribution function and then we discuss the general distributional change. Simulated critical values for the proposed tests are obtained and compared with Monte Carlo asymptotic quantiles of the corresponding limiting distributions. We also conduct Monte Carlo power comparisons of the proposed tests with six other tests which may also be used to test the same hypotheses. Finally, in Chapter 6 we give some concluding remarks and suggest some related future research problems.

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## Chapter 2

# Weighted tests for a slope change in simple regression

## 2.1 Introduction

Let $Y_1, \ldots, Y_n$ be a sequence of independent rv's generated over successive time points $0 < x_1 < x_2 < \ldots < x_n < \infty$. Under the null hypothesis, the $Y_i$ 's follow the model

$$Y_i = \alpha + \beta x_i + \epsilon_i \qquad , i = 1, \ldots, n, \tag{2.1}$$

where $\alpha$ and $\beta$ are unknown real numbers, and $\epsilon_1, \ldots, \epsilon_n$ are iid rv's with mean zero and finite variance $\sigma^2$. Without loss of generality we will assume that $\sigma^2 = 1$. Suppose now we are interested in testing whether or not there is a change in the regression slope $\beta$ at an unknown point $m$, $1 \leq m \leq n-1$. The problem then is to test if the regression model has changed to

$$Y_i = \alpha + \begin{cases} \beta x_i + \epsilon_i & \text{for } i = 1, \ldots, m \\ \gamma x_i + \epsilon_i & \text{for } i = m+1, \ldots, n, \end{cases} \tag{2.2}$$

where $m$ is unknown. The above change point problem is equivalent to testing:

$$H_o : \beta = \gamma \qquad \text{vs} \qquad H_1 : \beta \neq \gamma. \tag{2.3}$$

All the test statistics developed in this chapter are independent of the unknown true value of $\alpha$. For this reason we will assume without loss of generality that $\alpha = 0$.

Assume that $H_o$ is true. The least squares (LS) estimator of $\beta$ based on $x_1, \ldots, x_k$ is given by

$$\hat{\beta}_k = \sum_{i=1}^{k} (x_i - \overline{x}_k) Y_i / v_k^2, \qquad k = 2, \ldots, n, \tag{2.4}$$

where $\overline{x}_k = \sum_{i=1}^{k} x_i / k$, $v_k^2 = \sum_{i=1}^{k} (x_i - \overline{x}_k)^2$ and $\hat{\beta}_1 = 0$. Now consider the following differences

$$D_n(k) = \hat{\beta}_k - \hat{\beta}_n, \qquad k = 2, \ldots, n, \tag{2.5}$$

and note that

$$D_n(k) = \sum_{i=1}^{k} (x_i - \overline{x}_k) e_i / v_k^2 \qquad , k = 2, \ldots, n, \tag{2.6}$$

where $e_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_n x_i$, $i = 1, \ldots, n$.

Following Sen (1980), we assume that

$$\lim_{n \to \infty} \overline{x}_{[nt]} = \mu(t) \tag{2.7}$$

and

$$\lim_{n \to \infty} v_{[nt]}^2 / n = \eta(t)^2, \tag{2.8}$$

where $0 \leq t \leq 1$, and both $\mu(.)$ and $\eta(.)$ exist and are continuous on $[0, 1]$. Let $k_n(.)$ be the nondecreasing right-continuous integer-valued function given by

$$k_n(u) = \sup_{0 < s < 1} ([ns] : f_n(s) \leq u) \qquad , 0 < u < 1, \tag{2.9}$$

where $f_n(s) = v_{[ns]}^2/v_n^2$. It is clear that $f_n(.)$ is a non-decreasing function in $s \in (0,1)$. In addition we assume that

$$f_n(t)(\overline{x}_n/\overline{x}_{[nt]}) \leq 1 \qquad as \;\; t \uparrow 1. \tag{2.10}$$

It is easy to see that (2.10) is satisfied when $x_i = i/n$, $1 \leq i \leq n$, for all $t \in (0,1)$.

Sen (1980) studied the process $M_n(u)$, $0 < u < 1$ given by

$$M_n(u) = \widehat{M}(k_n(u)), \qquad 0 < u < 1, \tag{2.11}$$

where $\widehat{M}(1) = \widehat{M}(0) = 0$ ,

$$\widehat{M}(l) = (v_l^2/v_n)D_n(l), \qquad 2 \leq l \leq n-1 \tag{2.12}$$

and $v_l^2$ and $D_n(l), l = 2, \ldots, n-1$ are defined by (2.4) and (2.5), respectively. He proposed the Kolmogorov-Smirnov-type statistic

$$\tau_n = \max_{2 \leq l \leq n-1} \widehat{M}(l)$$

for testing $H_\circ$ . Sen (1980) proved the weak convergence of $M_n(.)$ to a Brownian bridge and used this result to obtain the limiting distribution of $\tau_n$ .

In this Chapter we propose and study Anderson-Darling and Erdős-Darling type statistics for testing $H_\circ$. To obtain the limiting distributions of the proposed test statistics we will prove that the process $M_n(.)$ converges in probability to a Brownian bridge in the sup-norm metric. Finally we introduce test statistics for the problem of testing against the epidemic and the at most two change points alternatives.

In section 2 the (in probability) convergence results of the process $M_n(.)$ and the asymptotic distribution theory of an Anderson-Darling type test are given. In section 3 we discuss the limiting distribution of an Erdős-Darling type test. Tests for the epidemic and the at most two change points alternatives are introduced in section 4. Finally in section 5 we report the results of a Monte Carlo simulation study for the critical values and powers of the above change point tests.

## 2.2 Convergence results

Define the process $\{S_n(s), 0 \leq s \leq 1\}$, as $S_n(0) = 0$ and

$$S_n(s) = \sum_{i=1}^{[ns]}(x_i - \overline{x}_{[ns]})Y_i, \qquad 0 < s \leq 1, n \geq 1. \tag{2.13}$$

Note that the process $\widehat{M}(.)$ of (2.12) can be expressed in terms of $S_n(.)$ as follows

$$\widehat{M}([ns]) = \begin{cases} (S_n(s) - f_n(s)S_n(1))/v_n & ,2/n \leq s \leq 1 - 1/n \\ 0 & ,s < 2/n, s > 1 - 1/n. \end{cases} \tag{2.14}$$

Let $S_\circ = 0$ and $S_j = \sum_{i=1}^{j} Y_i$, $j = 1, \ldots, n$, then we have;

$$\begin{aligned} S_n(s) &= \sum_{i=1}^{[ns]} x_i Y_i - \overline{x}_{[ns]} \sum_{i=1}^{[ns]} Y_i \\ &= \sum_{i=1}^{[ns]} x_i(S_i - S_{i-1}) - \overline{x}_{[ns]} S_{[ns]} \\ &= x_{[ns]} S_{[ns]} - \sum_{i=1}^{[ns]-1} (x_{i+1} - x_i) S_i - \overline{x}_{[ns]} S_{[ns]} \\ &= (x_{[ns]} - \overline{x}_{[ns]}) S_{[ns]} - \sum_{i=1}^{[ns]-1} (x_{i+1} - x_i) S_i. \end{aligned} \tag{2.15}$$

In our proofs we will need the following result of Major (1979).

**Theorem** (2.0)

Let a distribution $F(y)$ be given with $\int y dF(y) = 0$ and $\int y^2 dF(y) = 1$.

Define

$$\sigma_k^2 = \int_{-\sqrt{2^n}}^{\sqrt{2^n}} y^2 dF(y) - \left(\int_{-\sqrt{2^n}}^{\sqrt{2^n}} y dF(y)\right)^2, \qquad 2^n \le k < 2^{n+1}, n \ge 1.$$

A sequence of iid rv's $Y_1, Y_2, \ldots$ with distribution $F(.)$ and a sequence of independent normal rv's $Z_1^*, Z_2^*, \ldots$ with $E(Z_k^*) = 0$ and $E(Z_k^{*2}) = \sigma_k^2$ can be constructed in such a way that the partial sums $S_n = Y_1 + Y_2 + \ldots + Y_n$ and $T_n^* = Z_1^* + Z_2^* + \ldots + Z_n^*$, $n = 1, 2, \ldots$ satisfy the relation

$$| S_n - T_n^* | \overset{a.s.}{=} o(n^{1/2}).$$

Note that Theorem (2.0) implies that

$$| S_n - T_n^* | / \sqrt{n} \overset{a.s.}{=} o(1). \tag{2.16}$$

Csörgő and Révész (1981), p.112, showed that if we define a Wiener process $T_m = \sum_{i=1}^m Z_i$, where $Z_i = Z_i^* / \sigma_i$, $i \ge 1$ and the $Z_i^*$'s are as in Theorem (2.0), then

$$\sup_{0 \le s \le 1} | T_{[ns]}^* - T_{[ns]} | / \sqrt{n} \overset{P}{=} o(1). \tag{2.17}$$

From (2.16) and (2.17) we obtain

$$\begin{aligned} | S_n - T_n | / \sqrt{n} &\le | S_n - T_n^* | / \sqrt{n} + | T_n^* - T_n | / \sqrt{n} \\ &= o_p(1) + o_p(1) \\ &\overset{P}{=} o(1). \end{aligned} \tag{2.18}$$

**Corollary** (2.0)

The result of (2.18) implies

$$\sup_{1\leq m\leq n} \mid S_m - T_m \mid / \sqrt{m} \overset{\mathrm{p}}{=} O(1) \qquad \text{as } n \to \infty. \tag{2.19}$$

$$\sup_{1\leq m\leq n} \mid S_m - T_m \mid / \sqrt{n} \overset{\mathrm{p}}{=} o(1) \qquad \text{as } n \to \infty. \tag{2.20}$$

Define the Gaussian processes $\{W_n(s),\ 0 \leq s \leq 1\}$ and $\{B_n(s),\ 0 \leq s \leq 1\}$ by

$$W_n(s) = (x_{[ns]} - \overline{x}_{[ns]})T_{[ns]} - \sum_{i=1}^{[ns]-1} (x_{i+1} - x_i)T_i \quad ,0 < s \leq 1, \tag{2.21}$$

and

$$B_n(s) = \{(W_n(s) - f_n(s)W_n(1))/v_n\} I(2/n \leq s \leq 1 - 1/n), \tag{2.22}$$

where $I(A)$ is the indicator function of the set $A$. In Appendix A, we proved that

$$v_n^{-1} W_n(s) \overset{\mathrm{d}}{=} W_1(f_n(s)), \quad 0 < s \leq 1 \tag{2.23}$$

and

$$B_n(s) \overset{\mathrm{d}}{=} B_1(f_n(s)) \quad ,0 < s < 1, \tag{2.24}$$

where $v_n$ and $f_n(.)$ are as in (2.4), $W_1(.)$ is a standard Wiener process and $B_1(t) = W_1(t) - tW_1(1)$, $0 < t < 1$ is a standard Brownian bridge.

Next, we prove a convergence (in probability) result for the process $\widehat{M}(.)$ in (2.14).

**Theorem** (2.1)

Let $\widehat{M}(.)$ and $B_n(.)$ be as in (2.14) and (2.22), respectively. Then as n → ∞,

$$a_n = \sup_{0<t<1} \mid \widehat{M}([nt]) - B_n(t) \mid \overset{p}{=} o(1).$$

**Proof**

By (2.14) and (2.24) we have

$$a_n \leq \sup_{0<t<1} \mid S_n(t) - W_n(t) \mid /v_n + \sup_{0<t<1} f_n(t) \mid S_n(1) - W_n(1) \mid /v_n, \quad (2.25)$$

and since $f_n(t) \leq 1$, $\sqrt{n}/v_n = O(1)$ (by (2.8)), the required result will follow if we show that

$$a_{n1} = \sup_{0<t<1} \mid S_n(t) - W_n(t) \mid /\sqrt{n} \overset{p}{=} o(1). \quad (2.26)$$

Using the representation of $S_n(.)$ in (2.15) with (2.21) we have

$$\begin{aligned} a_{n1} &\leq n^{-1/2} \sup_{0<t<1} \{\mid x_{[nt]} - \overline{x}_{[nt]} \mid\mid S_{[nt]} - T_{[nt]} \mid + \sum_{i=1}^{[nt]-1} \mid x_{i+1} - x_i \mid\mid S_i - T_i \mid\} \\ &\leq n^{-1/2} \max_{1\leq m\leq n} \mid S_m - T_m \mid \{ \sup_{0<t<1} \mid x_{[nt]} - \overline{x}_{[nt]} \mid \\ &\qquad + \sup_{0<t<1} \mid x_{[nt]} - x_1 \mid\}. \quad (2.27) \end{aligned}$$

By (2.20) and the fact that both terms inside the bracket are bounded (see Appendix B), we obtain the required result.

Let $Q$ be the class of weight functions $q(t)$, $0 < t < 1$ such that

(i) $\inf_{\delta \leq t \leq 1-\delta} q(t) > 0, \qquad \delta \in (0, 1/2)$

(ii) $\lim_{t \downarrow 0} q(t)/t^{1/2} = \lim_{t \uparrow 1} q(t)/(1-t)^{1/2} = \infty$

(iii) $q(.)$ is non-decreasing in a neighbourhood of zero and non-increasing in a neighbourhood of one.

Given $q \in Q$ and $f_n : [0,1] \to [0,1]$ defined by (2.9) we assume that there exists $q_1 \in Q$ such that

$$q(f_n(t)) = q_1(t), \qquad 0 < t < 1. \tag{2.28}$$

Assumption (2.28) above is satisfied for all regressors $x_i$'s of the form $x_i = (i/n)^p$, $1 \leq i \leq n$, $p > 0$. This is because for $x_i = (i/n)^p$, $1 \leq i \leq n$, $p > 0$, we have $f_n(s) \approx s^{2p+1} = s^r$, $0 < s < 1$, $r = 2p+1$ and hence, for a weight function $q(t) = [t(1-t)]^{1/(2+\delta)r}$, $\delta > 0$, $r > 1$, we get $q(f_n(s)) \approx [s^r(1-s^r)]^{1/(2+\delta)r}$, that is , $q(f_n(s)) \in Q$.

**Theorem** (2.2)

Let $q \in Q$ be a weight function satisfying (2.28). Then, as $n \to \infty$

$$d_n = \sup_{0<t<1} \mid \widehat{M}([nt]) - B_n(t) \mid / q(f_n(t)) \overset{P}{=} o(1),$$

where $\widehat{M}(.)$ and $B_n(.)$ are as in Theorem (2.1).

**Proof**

Let,

$$G_n(t) = \{\widehat{M}([nt]) - B_n(t)\}/q(f_n(t)), \qquad 0 < t < 1.$$

Then,

$$d_n = \sup_{0<t<1} \mid G_n(t) \mid .$$

Now, for a fixed $\delta \in (0, \frac{1}{2})$ and $n$ such that $\delta > \frac{2}{n}$, we have a.s.

$$\begin{aligned} d_n \;\le\; & \sup_{0<t<\frac{2}{n}} \mid G_n(t) \mid + \sup_{\frac{2}{n} \le t < \delta} \mid G_n(t) \mid + \sup_{\delta \le t \le 1-\delta} \mid G_n(t) \mid \\ & + \sup_{1-\delta < t \le 1-\frac{1}{n}} \mid G_n(t) \mid + \sup_{1-\frac{1}{n} < t < 1} \mid G_n(t) \mid \\ = \; & a_1(n) + a_2(n) + a_3(n) + a_4(n) + a_5(n). \qquad (2.29) \end{aligned}$$

The first and last terms of (2.29) above are equal to zero by the definition of the processes in (2.14) and (2.22). Since $q(.)$ is bounded away from zero inside $(\delta, 1-\delta)$, (see condition (i) of $Q$), then by Theorem (2.1), the third term is

$$a_3(n) \overset{P}{=} o(1). \qquad (2.30)$$

Since, $\sqrt{n} v_n^{-1} = O(1)$ we have

$$\begin{aligned} a_2(n) \le O(1)\{ \quad & n^{-\frac{1}{2}} \sup_{\frac{2}{n} \le t < \delta} \mid S_n(t) - W_n(t) \mid / q(f_n(t)) \\ + \quad & n^{-\frac{1}{2}} \sup_{\frac{2}{n} \le t < \delta} \mid S_n(1) - W_n(1) \mid f_n(t)/q(f_n(t))\}. \quad (2.31) \end{aligned}$$

Using Corollary(2.0) for the first term of (2.31) and the proof of Theorem (2.1) for the second term we obtain as $n \to \infty$

$$\begin{aligned} a_2(n) \le O(1)\{ \quad & (nt)^{-\frac{1}{2}} \sup_{\frac{2}{n} \le t < \delta} \mid S_n(t) - W_n(t) \mid \sup_{\frac{2}{n} \le t < \delta} t^{\frac{1}{2}} / q_1(t) \\ + \quad & n^{-\frac{1}{2}} \mid S_n(1) - W_n(1) \mid \sup_{\frac{2}{n} \le t < \delta} f_n(t)/q(f_n(t))\} \\ = O(1)\{ \quad & O_p(1) \sup_{\frac{2}{n} \le t < \delta} t^{\frac{1}{2}} / q_1(t) + o_p(1) \sup_{\frac{2}{n} \le t < \delta} f_n(t)/q(f_n(t))\}. (2.32) \end{aligned}$$

Taking $\delta > 0$ arbitrarily small we get by condition (ii) of $Q$,

$$a_2(n) \overset{\mathrm{P}}{=} o(1). \tag{2.33}$$

Next, we show that the fourth term of (2.29) converges to zero in probability.

$$\begin{aligned}
a_4(n) = (n^{\frac{1}{2}} v_n^{-1}) \quad & n^{-\frac{1}{2}} \sup_{1-\delta<t\leq 1-\frac{1}{n}} \{\mid S_n(t) - W_n(t) - f_n(t)(S_n(1) \\
& -W_n(1)) \mid / q(f_n(t))\} \\
= O(1) \quad & n^{-\frac{1}{2}} \sup_{1-\delta<t\leq 1-\frac{1}{n}} \{\mid \sum_{i=1}^{n} (x_i - \overline{x}_{[nt]}) Y_i - \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) Y_i \\
- \quad & (\sum_{i=1}^{n} (x_i - \overline{x}_{[nt]}) Z_i - \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) Z_i) \\
- \quad & f_n(t)(S_n(1) - W_n(1)) \mid / q(f_n(t))\} \\
\leq O(1)\{ \quad & n^{-\frac{1}{2}} \sup_{1-\delta<t\leq 1-\frac{1}{n}} \mid \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) Y_i - \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) \\
.Z_i \mid \; / q(f_n(t)) \quad & +n^{-\frac{1}{2}} \sup_{1-\delta<t\leq 1-\frac{1}{n}} \mid \sum_{i=1}^{n} (x_i - \overline{x}_{[nt]}) Y_i - \sum_{i=1}^{n} (x_i - \overline{x}_{[nt]}) Z_i \\
- \quad & f_n(t)(S_n(1) - W_n(1)) \mid / q(f_n(t))\} \\
= O(1)\{I_1(n) \quad + \quad & I_2(n)\}.
\end{aligned} \tag{2.34}$$

First, we observe that

$$\begin{aligned}
I_1(n) &= n^{-\frac{1}{2}} \sup_{1-\delta<t\leq 1-\frac{1}{n}} \mid \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) Y_i - \sum_{i=[nt]+1}^{n} (x_i - \overline{x}_{[nt]}) Z_i \mid / q(f_n(t)) \\
&= n^{-\frac{1}{2}} \sup_{\frac{1}{n}<t\leq\delta} \mid \sum_{i=[n(1-t)]+1}^{n} (x_i - \overline{x}_{[n(1-t)]}) Y_i - \sum_{i=[n(1-t)]+1}^{n} (x_i - \overline{x}_{[n(1-t)]}) Z_i \mid \\
&\quad / q_1(1-t) \\
&\leq \{ \sup_{\frac{1}{n}<t\leq\delta} \mid \sum_{j=1}^{n-[n(1-t)]} (x_{n-j+1} - \overline{x}_{[n(1-t)]}) Y_{n-j+1} - \sum_{j=1}^{n-[n(1-t)]} (x_{n-j+1} - \overline{x}_{[n(1-t)]}) \\
&\quad .Z_{n-j+1} \mid / \sqrt{n - [n(1-t)]}\} \cdot \sup_{\frac{1}{n}<t\leq\delta} \frac{(n - [n(1-t)])^{\frac{1}{2}}}{n^{\frac{1}{2}} q_1(1-t)}
\end{aligned}$$

$$\begin{aligned}
&\le \{\max_{1\le m\le n} \mid \sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Y_{n-j+1}-\sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Z_{n-j+1}\mid / \\
&\quad \sqrt{m}\}.\{\sup_{\frac{1}{n}<t\le\delta}\frac{(1-\frac{(n(1-t)-1)}{n})^{\frac{1}{2}}}{q_1(1-t)}\} \\
&\le \{\max_{1\le m\le n} \mid \sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Y_{n-j+1}-\sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Z_{n-j+1}\mid / \\
&\quad \sqrt{m}\}\cdot\{\sup_{\frac{1}{n}<t\le\delta}\frac{(t+\frac{1}{n})^{\frac{1}{2}}}{q_1(1-t)}\} \\
&\le \{\max_{1\le m\le n} \mid \sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Y_{n-j+1}-\sum_{j=1}^{m}(x_{n-j+1}-\overline{x}_{(n-m)})Z_{n-j+1}\mid / \\
&\quad \sqrt{m}\}\{\sup_{\frac{1}{n}<t\le\delta}\frac{(t+t)^{\frac{1}{2}}}{q_1(1-t)}\}. \qquad (2.35)
\end{aligned}$$

For the first factor of (2.35) we use the same argument used in $a_2(.)$ to obtain

$$I_1(n)=O_p(1).\sup_{\frac{1}{n}<t\le\delta}\frac{t^{\frac{1}{2}}}{q_1(1-t)}. \qquad (2.36)$$

Hence for arbitrarily small $\delta>0$ we have as $n\to\infty$

$$I_1(n)\overset{\mathrm{p}}{=}o(1). \qquad (2.37)$$

By Abel's summation we have

$$\begin{aligned}
&\sum_{i=1}^{n}(x_i-\overline{x}_{[nt]})Y_i-f_n(t)\sum_{i=1}^{n}(x_i-\overline{x}_n)Y_i \\
=\;&(x_n-\overline{x}_{[nt]})S_n-\sum_{i=1}^{n-1}(x_{i+1}-x_i)S_i-f_n(t)\{(x_n-\overline{x}_n)S_n-\sum_{i=1}^{n-1}(x_{i+1}-x_i)S_i\} \\
=\;&(1-f_n(t)).\{x_nS_n-\sum_{i=1}^{n-1}(x_{i+1}-x_i)S_i\}-S_n\overline{x}_{[nt]}(1-f_n(t)\frac{\overline{x}_n}{\overline{x}_{[nt]}}) \qquad (2.38)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{n}(x_i-\overline{x}_{[nt]})Z_i-f_n(t)\sum_{i=1}^{n}(x_i-\overline{x}_n)Z_i \\
=\;&(1-f_n(t))\{x_nT_n-\sum_{i=1}^{n-1}(x_{i+1}-x_i)T_i\}-T_n\overline{x}_{[nt]}(1-f_n(t)\frac{\overline{x}_n}{\overline{x}_{[nt]}}), \qquad (2.39)
\end{aligned}$$

where, $S_j = \sum_{i=1}^{j} Y_i$ and $T_j = \sum_{i=1}^{j} Z_i$, $1 \leq j \leq n$. By substituting (2.38) and (2.39) in $I_2(.)$ of (2.34) we get

$$\begin{aligned} I_2(n) \leq n^{-\frac{1}{2}} & \sup_{1-\delta<t<1} \mid x_n(1 - f_n(t))(S_n - T_n) \mid /q(f_n(t)) \\ & + \; n^{-\frac{1}{2}} \sup_{1-\delta<t<1} \mid (1 - f_n(t)) \sum_{i=1}^{n-1} (x_{i+1} - x_i)(S_i - T_i) \mid /q(f_n(t)) \\ & + \; n^{-\frac{1}{2}} \sup_{1-\delta<t<1} \mid \overline{x}_{[nt]}(1 - f_n(t) \frac{\overline{x}_n}{\overline{x}_{[nt]}})(S_n - T_n) \mid /q(f_n(t)) \\ & = \; J_1(n) + J_2(n) + J_3(n). \end{aligned} \tag{2.40}$$

By (2.20) we have

$$\begin{aligned} J_1(n) &= x_n\{\mid S_n - T_n \mid /\sqrt{n}\} \sup_{1-\delta<t<1} \frac{(1 - f_n(t))}{q(f_n(t))} \\ &= O(1) \, o_p(1) \sup_{1-\delta<t<1} \frac{(1 - f_n(t))}{q(f_n(t))} \end{aligned} \tag{2.41}$$

and

$$\begin{aligned} J_2(n) &\leq \{\max_{1 \leq i \leq n} \mid S_i - T_i \mid /\sqrt{n}\}(x_n - x_1) \sup_{1-\delta<t<1} \frac{(1 - f_n(t))}{q(f_n(t))} \\ &= o_p(1)O(1) \sup_{1-\delta<t<1} \frac{(1 - f_n(t))}{q(f_n(t))}. \end{aligned} \tag{2.42}$$

By (2.10) and (2.20) we have

$$\begin{aligned} J_3(n) &\leq \{\mid S_n - T_n \mid /\sqrt{n}\} \sup_{1-\delta<t<1} \overline{x}_{[nt]}\{\frac{(1 - f_n(t)\frac{\overline{x}_n}{\overline{x}_{[nt]}})}{q(f_n(t))}\}, \\ &= o_p(1)O(1) \sup_{1-\delta<t<1} \frac{(1 - f_n(t))}{q(f_n(t))}. \end{aligned} \tag{2.43}$$

Taking $\delta > 0$ arbitrarily small, we obtain

$$J_1(n) = J_2(n) = J_3(n) \overset{\mathrm{P}}{=} o(1). \tag{2.44}$$

By (2.34), (2.37), (2.40) and (2.44) we have

$$a_4(n) \overset{\mathrm{P}}{=} o(1). \tag{2.45}$$

Thus by (2.29), (2.30), (2.33), (2.45) and the fact that the first and last terms of (2.29) are zeros, the proof of the Theorem is complete.

Next we obtain the limiting distribution of the Anderson-Darling type test statistic. First, we note that for the function $f_n(.)$ of (2.9)we have

$$I_f = \int_0^1 \{f_n(t)(1 - f_n(t))\}^{-3/4} \mathrm{d} f_n(t) < \infty, \tag{2.46}$$

for arbitrarily large $n$.

**Corollary** (2.1)

$$A_n^2 = \int_0^1 \{\frac{\widehat{M}([nt])}{\sqrt{f_n(t)(1 - f_n(t))}}\}^2 \mathrm{d} f_n(t) \xrightarrow{\mathrm{P}} L_n^2 = \int_0^1 \{\frac{B_n(t)}{\sqrt{f_n(t)(1 - f_n(t))}}\}^2 \mathrm{d} f_n(t),$$

where $\widehat{M}(.)$ and $B_n(.)$ are defined by (2.14) and (2.22) respectively.

**Proof**

Let $q(t) = \{t(1-t)\}^{\frac{1}{4}}$. It is clear that $q \in Q$ and by (2.46) and Theorems (2.1) and (2.2) we have

$$\begin{aligned} \mid A_n^2 - L_n^2 \mid \; &\leq \; \int_0^1 \frac{\mid \widehat{M}^2([nt]) - B_n^2(t) \mid}{f_n(t)(1 - f_n(t))} \mathrm{d} f_n(t) \\ &\leq \; I_f \sup_{0<t<1} \frac{\mid \widehat{M}^2([nt]) - B_n^2(t) \mid}{q(f_n(t))} \\ &\leq \; I_f \{\sup_{0<t<1} \frac{\mid \widehat{M}([nt]) - B_n(t) \mid}{q(f_n(t))}\}\{\sup_{0<t<1} (\mid \widehat{M}([nt]) - \\ &\quad\; B_n(t) \mid +2 \mid B_n(t) \mid)\} \end{aligned}$$

$$\begin{aligned} &\le O(1)\, o_p(1)\{o_p(1) + O_p(1)\} \\ &\overset{\mathrm{P}}{=} o(1). \end{aligned} \tag{2.47}$$

By (2.24) we have

$$L_n^2 \overset{\mathrm{d}}{=} \tilde{L}_n^2 = \int_0^1 \{\frac{B_1(f_n(t))}{\sqrt{f_n(t)(1-f_n(t))}}\}^2 \mathrm{d} f_n(t). \tag{2.48}$$

Now in (2.48), let $t = f_n^{-1}(u) = \sup_{0<x<1}\{x : f_n(x) \le u\}$ , then

$$\begin{aligned} \tilde{L}_n^2 &= \int_0^1 \{\frac{B_1(f_n f_n^{-1}(u))}{\sqrt{f_n f_n^{-1}(u)(1 - f_n f_n^{-1}(u))}}\}^2 \mathrm{d} f_n f_n^{-1}(u) \\ &= \int_0^1 \{\frac{B_1(u_n)}{\sqrt{u_n(1-u_n)}}\}^2 \mathrm{d}u_n, \end{aligned} \tag{2.49}$$

where $u_n = f_n f_n^{-1}(u) \le u$ and $u_n \to u$ as $n \to \infty$ . Using the principle component decomposition (see Shorack and Wellner (1986)), we have

$$\begin{aligned} R(u_n) &= \frac{B_1(u_n)}{\sqrt{u_n(1-u_n)}} \\ &\overset{\mathrm{d}}{=} \sum_{j=1}^{\infty} \sqrt{\lambda_j} N_j \phi_j(u_n) \\ &= 2 \sum_{j=1}^{\infty} \frac{N_j}{\sqrt{j(j+1)}} \sqrt{\frac{2j+1}{j(j+1)}} \sqrt{u_n(1-u_n)}\, P_j^{'}(2u_n - 1), \end{aligned} \tag{2.50}$$

where $\lambda_j = \frac{1}{j(j+1)}$, $N_j, j = 1, \ldots$ are iid $N(0,1)$ rv's and for $0 \le \theta \le 1$,

$$\phi_j(\theta) = 2\sqrt{\frac{2j+1}{j(j+1)}} \sqrt{\theta(1-\theta)}\, P_j^{'}(2\theta - 1),$$

with $P_j^{'}(.)$ is the derivative of the Legendre polynomial of degree $j$, $\int_0^1 \phi_j^2(\theta)\mathrm{d}\theta = 1$ and $\int_0^1 \phi_j(\theta)\phi_i(\theta)\mathrm{d}\theta = 0$, $i \ne j$. By the boundedness of $\phi_j(.)$ and the result that $u_n \to u$ as $n \to \infty$ we have $\int_0^1 \phi_j^2(u_n)\mathrm{d}u_n = 1$ and $\int_0^1 \phi_j(u_n)\phi_i(u_n)\mathrm{d}u_n = 0, i \ne j$

as $n \to \infty$. Hence by (2.49) and (2.50) the series decomposition of $\tilde{L}_n^2$ converges to the series decomposition of $A^2$ ( the Anderson-Darling limiting rv ) given by

$$\begin{aligned} A^2 &= \int_0^1 \{\frac{B_1(u)}{\sqrt{u(1-u)}}\}^2 \mathrm{d}u \\ &= \sum_{j=1}^{\infty} N_j^2/j(j+1). \end{aligned}$$

This implies that

$$\tilde{L}_n^2 \xrightarrow{\mathrm{d}} A^2. \tag{2.51}$$

By Corollary (2.1), (2.48) and (2.51) we have, $A_n^2 \xrightarrow{\mathrm{d}} A^2$, as $n \to \infty$.

## 2.3 On the Erdős-Darling type test

Let $\widehat{M}(.)$ be as defined in (2.14). In this section we will discuss the limiting distribution of the following test statistic,

$$\begin{aligned} E_n &= \sup_{\frac{1}{n} \le f_n(t) \le 1-\frac{1}{n}} \frac{\mid \widehat{M}([nt]) \mid}{\sqrt{f_n(t)(1-f_n(t))}} \\ &= \sup_{\frac{1}{n} \le u \le 1-\frac{1}{n}} \frac{\mid M_n^*(u) \mid}{\sqrt{u(1-u)}}, \end{aligned} \tag{2.52}$$

where $M_n^*(u) = \widehat{M}([nf_n^{-1}(u)])$. To investigate such limiting theory we need first to prove some additional results. We will assume throughout the rest of this section that, under $H_o$, the rv's $Y_1, \ldots, Y_n$ of (2.1) are iid mean zero, variance one and that $E(e^{\theta Y_1})$ exists in a neighbourhood of $\theta = 0$. Under these assumptions, Komlós, Major and Tusnády (1976)'s strong approximation result implies that there exists a Wiener process $W(.)$ such that

$$\max_{1 \leq k \leq n} \mid S_k - W(k) \mid \overset{\text{a.s}}{=} O(\log\, n), \tag{2.53}$$

where $S_k = Y_1 + \ldots + Y_k$ and $S_o = 0$. Define the Gaussian process $\{G_n(s), 0 \leq s \leq 1\}$ by $G_n(0) = 0$ and:

$$G_n(s) = (x_{[ns]} - \overline{x}_{[ns]})W([ns]) - \sum_{i=1}^{[ns]-1} (x_{i+1} - x_i)W(i), \tag{2.54}$$

where $W(.)$ is the Wiener process of (2.53). By the definition of $G_n(.)$ above we see that

$$v_n^{-1} G_n(s) \overset{\text{d}}{=} W_1(f_n(s)), \tag{2.55}$$

where $v_n^2$ and $f_n(.)$ are as in (2.4), and $W_1(.)$ is a standard Wiener process. Next we define the Gaussian process $\{B_n(s), 0 < s < 1\}$ by

$$B_n(s) = v_n^{-1}\{G_n(s) - f_n(s)G_n(1)\}. \tag{2.56}$$

Clearly $B_n(s) \overset{\text{d}}{=} B_1(f_n(s))$, where $B_1(s) = W_1(s) - sW_1(1)$ is a Brownian bridge.

We now introduce a strong approximation result for the process $\widehat{M}(.)$ of (2.14).

**Theorem** (2.3)

Assume that the conditions of (2.53) hold, then

$$\sup_{0<s<1} \mid \widehat{M}([ns]) - B_n(s) \mid \overset{\text{a.s}}{=} O(n^{-\frac{1}{2}} \log n) \tag{2.57}$$

**Proof**

We have a.s.

$$\begin{aligned}\sup_{0<s<1} \mid \widehat{M}([ns]) - B_n(s) \mid \ \leq \ & \sup_{0<s<1} \mid v_n^{-1}(S_n(s) - G_n(s)) \mid \\ & + \sup_{0<s<1} f_n(s) \mid v_n^{-1}(S_n(1) - G_n(1)) \mid .\end{aligned}$$

Since, $\sup_{0<s<1} f_n(s) = 1$ and by (2.8), $\lim_{n\to\infty} \frac{\sqrt{n}}{v_n} = O(1)$ , then the required result will follow if we show that

$$C_n = \sup_{0<s<1} \mid S_n(s) - G_n(s) \mid \overset{\text{a.s}}{=} O(\log n). \tag{2.58}$$

Using the expression of $S_n(.)$ and the definition of $G_n(.)$ it is easy to see that

$$\begin{aligned} C_n \ & \leq \ \sup_{0<s<1} \{ \mid x_{[ns]} - \overline{x}_{[ns]} \mid \cdot \mid S_n - W([ns]) \mid + \sum_{i=1}^{[ns]-1} (x_{i+1} - x_i) \mid S_i - W(i) \mid \} \\ & \leq \ \max_{1\leq k\leq n} \mid S_k - W(k) \mid \{ \sup_{0<s<1} \mid x_{[ns]} - \overline{x}_{[ns]} \mid + \sup_{0<s<1} \mid x_{[ns]} - x_1 \mid \}. \end{aligned}$$

Hence by (2.53) we get

$$\begin{aligned} C_n & \overset{\text{a.s}}{=} \ O(\log n)\{O(1) + O(1)\} \\ & \overset{\text{a.s}}{=} \ O(\log n). \end{aligned}$$

**Lemma** (2.1)

Let $W(.)$ be the Wiener process in (2.53), then

$$\sup_{\lambda \leq nt < n} \mid S_n(t) - G_n(t) \mid / (nt)^{\frac{1}{2}-r} \overset{\text{a.s}}{=} O(1)$$

for all $0 \leq r < \frac{1}{2}$ and $0 < \lambda < \infty$.

This Lemma is analogous to the Lemma of Csörgő and Horváth (1986) and it follows immediately from (2.58).

Let $Q_1$ be the class of weight functions $q(t)$, $0 < t < 1$, such that

(i) $\inf_{\delta \leq t \leq 1-\delta} q(t) > 0, \quad \delta \in (0, \frac{1}{2})$.

(ii) $\lim_{t \downarrow 0} t^{\frac{1}{2}-r}/q(t) = \lim_{t \uparrow 1} (1-t)^{\frac{1}{2}-r}/q(t) = O(1)\ , \quad 0 \leq r < (\frac{1}{2} - \frac{1}{g}),\ \ g > 2.$

(iii) $q(.)$ is non-decreasing in the neighbourhood of zero and non-increasing in the neighbourhood of one.

Given $q \in Q_1$ and $f_n : [0,1] \rightarrow [0,1]$, assume that there exists $q \in Q_1$ such that

$$q(f_n(s)) = q_1(s), \qquad 0 < s < 1, \tag{2.59}$$

where $f_n(.)$ is defined by (2.10). In the following Theorem we consider

$$q(t) = \{t(1-t)\}^{\frac{1}{2}-r},\ \ 0 < t < 1,\ \ 0 \leq r < (\frac{1}{2} - \frac{1}{g}), \tag{2.60}$$

for some $g > 2$. It is easy to see that $q(.) \in Q_1$.

**Theorem** (2.4)

Let $q \in Q_1$ and assume that (2.59) holds. Then under the conditions of Theorem (2.3) we have, as $n \to \infty$

$$n^r \sup_{0<t<1} \mid \widehat{M}([nt]) - B_n(t) \mid q(f_n(t)) \overset{\text{P}}{=} O(1),$$

where $0 \leq r < (\frac{1}{2} - \frac{1}{g})$, for some $g > 2$.

**Proof**

The required result follows by making minor changes in the proof of Theorem (2.2) ( e.g., replacing terms like $\sup_{\frac{2}{n} \leq t < \delta} t^{\frac{1}{2}}/q_1(t) = o(1)$, by $\sup_{\frac{2}{n} \leq t < \delta} t^{\frac{1}{2}-r}/q_1(t) = O(1)$, as $\delta \downarrow 0$ and $n \to \infty$).

Let $a(z) = (2 \log z)^{\frac{1}{2}}$, $b(z) = 2 \log z + \frac{1}{2} \log \log z - \frac{1}{2} \log \pi$, $a_n = a(\log n)$ and $b_n = b(\log n)$. The distribution function of the extreme value distribution is given by $E(s) = \exp\{-\exp(-s)\}$, $-\infty < s < \infty$.

**Theorem** (2.5)

For any $-\infty < x < \infty$, as $n \to \infty$ we have

$$P\{a_n E_n - b_n \leq x\} \to E^2(x),$$

where $E_n$ is defined by (2.52).

To prove this Theorem we need to state and prove some intermediate results.

**Lemma** (2.A): (Csörgő *et al.* (1986))

Let $\varepsilon_n = (\log n)^3/n$. For any $-\infty < x < \infty$ and as $n \to \infty$, we have

$$P\{a_n \sup_{\varepsilon_n \leq t \leq 1-\varepsilon_n} \mid B(t) \mid /(t(1-t))^{\frac{1}{2}} - b_n \leq x\} \to E^2(x),$$

where $B(.)$ is a Brownian bridge.

**Lemma** (2.2)

For any $-\infty < x < \infty$ and as $n \to \infty$, we have

$$P\{a_n \sup_{\varepsilon_n \leq f_n(t) \leq 1-\varepsilon_n} \mid B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - b_n \leq x\} \to E^2(x),$$

where $B_n(.)$ is defined by (2.56).

**Proof**:

By (2.56) and Lemma (2.A) we have as $n \to \infty$

$$\begin{aligned} & P\{a_n \sup_{\varepsilon_n \leq f_n(t) \leq 1-\varepsilon_n} \mid B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - b_n \leq x\} \\ = \; & P\{a_n \sup_{\varepsilon_n \leq f_n(t) \leq 1-\varepsilon_n} \mid B_1(f_n(t)) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - b_n \leq x\} \\ = \; & P\{a_n \sup_{\varepsilon_n \leq u \leq 1-\varepsilon_n} \mid B_1(u) \mid /(u(1-u))^{\frac{1}{2}} - b_n \leq x\} \to E^2(x). \end{aligned}$$

**Lemma** (2.3)

For any $-\infty < x < \infty$, and as $n \to \infty$, we have

$$P\{a_n \sup_{\varepsilon_n \leq f_n(t) \leq 1-\varepsilon_n} \mid \widehat{M}(\,[nt]) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - b_n \leq x\} \to E^2(x),$$

where $\widehat{M}(.)$ is defined by (2.14).

**Proof:**

Choose any $0 < r < \frac{1}{4}$. By Theorem (2.4) we have as $n \to \infty$

$$\begin{aligned} & a_n \sup_{\varepsilon_n \le f_n(t) \le 1-\varepsilon_n} \mid \widehat{M}(\,[nt]) - B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} \\ \le\ & \{ \sup_{\varepsilon_n \le f_n(t) \le 1-\varepsilon_n} \mid \widehat{M}(\,[nt]) - B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - r \\ & a_n \sup_{\varepsilon_n \le f_n(t) \le 1-\varepsilon_n} \{f_n(t)(1-f_n(t))\}^{-r}\} \\ \le\ & \{ \sup_{0<t<1} \mid \widehat{M}([nt]) - B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - r \\ & a_n \sup_{\varepsilon_n \le t \le 1-\varepsilon_n} \{t(1-t)\}^{-r}\} \\ \le\ & a_n(\frac{n}{\log^3 n})^r \sup_{0<t<1} \mid \widehat{M}([nt]) - B_n(t) \mid /(f_n(t)(1-f_n(t)))^{\frac{1}{2}} - r \\ =\ & O_p(\frac{a_n}{\log^{3r} n}) \overset{\text{P}}{=} o(1). \end{aligned}$$

Hence the proof of the Lemma follows from Lemma (2.2) and the above convergence.

**Lemma** (2.B): (Csörgő *et al.* (1986))

Let $\{\delta_n\}$ be any sequence of positive numbers such that $1 \le \delta_n \le n$, $\delta_n \to \infty$ and $(\delta_n/n) \to 0$, as $n \to \infty$. Let $B^*(t) = B(t)/(t(1-t))^{\frac{1}{2}}$, $0 < t < 1$. Then for any Brownian bridge $B(.)$, every $0 < c < \infty$ and $-\infty < x < \infty$ we have as $n \to \infty$,

$$P\{a(\log \delta_n) \sup_{\frac{c}{n} \le t \le \frac{\delta_n}{n}} \mid B^*(t) \mid - b(\log \delta_n^{\frac{1}{2}}) \le x\} \to E^2(x)$$

and

$$P\{a(\log\, \delta_n) \sup_{1-\frac{\delta_n}{n} \le t \le 1-\frac{c}{n}} \mid B^*(t) \mid - b(\log\, \delta_n^{\frac{1}{2}}) \le x\} \to E^2(x).$$

**Lemma** (2.4)

For any $\delta_n$ and $c$ as in Lemma (2.B), we have as $n \to \infty$

$$P\{a(\log\, \delta_n) \sup_{\frac{c}{n} \le f_n(t) \le \frac{\delta_n}{n}} \frac{\mid B_n(t) \mid}{\{f_n(t)(1-f_n(t))\}^{\frac{1}{2}}} - b(\log\, \delta_n^{\frac{1}{2}}) \le x\} \to E^2(x)$$

and

$$P\{a(\log\, \delta_n) \sup_{1-\frac{\delta_n}{n} \le f_n(t) \le 1-\frac{c}{n}} \frac{\mid B_n(t) \mid}{\{f_n(t)(1-f_n(t))\}^{\frac{1}{2}}} - b(\log\, \delta_n^{\frac{1}{2}}) \le x\} \to E^2(x),$$

where $-\infty < x < \infty$ and $B_n(.)$ is defined by (2.56).

The proof of this Lemma is exactly the same as in the proof of Lemma (2.2).

**Lemma** (2.5)

Let $\{\delta_n\}$ be any sequence of positive numbers such that $1 \le \delta_n \le n$, $\delta_n \to \infty$ and $(\delta_n/n) \to 0$, as $n \to \infty$.Then as $n \to \infty$ we have

$$m_1 = \sup_{\frac{1}{n} < f_n(t) \le \frac{\delta_n}{n}} \frac{\mid \widehat{M}([nt]) \mid}{\{f_n(t)(1-f_n(t)\}^{\frac{1}{2}} a(\log\, \delta_n)} \to_{\text{P}} 1 \qquad (2.61)$$

and

$$m_2 = \sup_{1-\frac{\delta_n}{n} < f_n(t) \le 1-\frac{1}{n}} \frac{\mid \widehat{M}([nt]) \mid}{\{f_n(t)(1-f_n(t)\}^{\frac{1}{2}} a(\log\, \delta_n)} \to_{\text{P}} 1. \qquad (2.62)$$

**Proof:**

As in Csörgő *et al.* (1986) we need only to prove (2.61) and (2.62) is proved similarly. By Theorem (2.4) we have as $n \to \infty$

$$\begin{aligned} &\left| \; m_1 - \sup_{\frac{1}{n} < f_n(t) \le \frac{\delta_n}{n}} \frac{| B_n(t) |}{\{f_n(t)(1 - f_n(t)\}^{\frac{1}{2}} a(\log \delta_n)} \right| \\ \le \; & n^{\tau} \sup_{0<t<1} \frac{| \widehat{M}([nt]) - B_n(t) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}-\tau}} (a(\log \delta_n))^{-1} \\ = \; & O_{\mathrm{P}}((a(\log \delta_n))^{-1}) \\ \overset{\mathrm{P}}{=} \; & \mathrm{o}(1). \end{aligned} \tag{2.63}$$

The first statement of Lemma (2.4) with $c = 1$ implies that as $n \to \infty$,

$$\sup_{\frac{1}{n} \le f_n(t) \le \frac{\delta_n}{n}} \frac{| B_n(t) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}} a(\log \delta_n)} \to_{\mathrm{P}} 1. \tag{2.64}$$

Hence (2.63) and (2.64) imply (2.61).

**Proof of Theorem (2.5):**

Consider the following rv's,

$$T_n := a_n \sup_{\frac{1}{n} \le f_n(t) \le 1 - \frac{1}{n}} \frac{| \widehat{M}([nt]) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}}} - b_n,$$

$$T_n^{(1)} := a_n \sup_{\frac{1}{n} \le f_n(t) \le \varepsilon_n} \frac{| \widehat{M}([nt]) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}}} - b_n,$$

$$T_n^{(2)} := a_n \sup_{\varepsilon_n \le f_n(t) \le 1 - \varepsilon_n} \frac{| \widehat{M}([nt]) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}}} - b_n$$

and

$$T_n^{(3)} := a_n \sup_{1 - \varepsilon_n \le f_n(t) \le 1 - \frac{1}{n}} \frac{| \widehat{M}([nt]) |}{\{f_n(t)(1 - f_n(t))\}^{\frac{1}{2}}} - b_n.$$

Then,

$$T_n = \max\{T_n^{(1)}, T_n^{(2)}, T_n^{(3)}\}. \tag{2.65}$$

By (2.65) and Lemma (2.3) we see that Theorem (2.5) will follow if we show that:

$$T_n^{(1)} \xrightarrow{\text{P}} -\infty, \qquad as\ n \to \infty \tag{2.66}$$

and

$$T_n^{(3)} \xrightarrow{\text{P}} -\infty, \qquad as\ n \to \infty. \tag{2.67}$$

Consider (2.66) and note that by Lemma (2.5) we have as $n \to \infty$;

$$T_n^{(1)} = a_n O_p(a(\log \delta_n)) - b_n \to -\infty.$$

A similar result holds for $T_n^{(3)}$.

## 2.4 Testing against the epidemic and the two change points alternatives

In this section we propose test statistics for two different change point problems. The first is the problem of testing against the epidemic (square) alternative in the slope of a simple regression model. This type of alternative was first introduced and studied by Levin and Kline (1985) in the context of testing for an epidemic change in a Binomial parameter. The second is the problem of testing against the at-most two change points alternative (AMTC) in the slope of a simple regression model.

### 2.4.1 Testing against the epidemic alternative

Consider the model (2.1) representing the null hypothesis of no change in the regression model against the epidemic (square) alternative which may be described by the model:

$$Y_i = \alpha + \begin{cases} \beta x_i + \epsilon_i & \text{for } i = 1, \ldots, k, l+1, \ldots, n \\ \gamma x_i + \epsilon_i & \text{for } i = k+1, \ldots, l \end{cases} , \qquad (2.68)$$

where the regression parameters $\alpha, \beta$ and $\gamma$ and the change points $k$ and $l$ are assumed to be unknown. The above testing problem is equivalent to:

$$H_\circ : \beta = \gamma \quad \text{vs} \qquad H_1 : \beta < \gamma \quad \text{or} \quad H_2 : \beta \neq \gamma. \qquad (2.69)$$

Define the process $\{M_n(u,v), 0 < u < v < 1\}$ by:

$$M_n(u,v) = M_n(v) - M_n(u), \qquad 0 < u < v < 1, \qquad (2.70)$$

where $M_n(.)$ is as defined by (2.11). Observe that the process in (2.70) can be written also as follows:

$$\begin{aligned} M_n(u,v) &= M(k_n(u), k_n(v)) \\ &= \widehat{M}(k_n(v)) - \widehat{M}(k_n(u)), \qquad 1 < k_n(u) < k_n(v) < n, (2.71) \end{aligned}$$

where $\widehat{M}(k_n(.))$ is as in (2.11). Note that the expected value of (2.71) is zero under $H_\circ$ of (2.69) (using the LS estimator properties), however the same expected value under the alternative $\overline{H}_\circ \in (H_1, H_2)$ is,

$$E(M(k,l)|\overline{H}_\circ) = \frac{(\gamma - \beta)}{v_n} \sum_{i=k+1}^{l} x_i\{(x_i - \overline{x}_l) - \Delta(x_i - \overline{x}_n)\}, \qquad (2.72)$$

where $\Delta = (v_l^2 - v_k^2)/v_n^2$ and $\overline{x}_m$ and $v_m^2$ are defined by (2.4).We can see that,

$$\sum_{i=k+1}^{l} x_i\{(x_i - \overline{x}_l) - \Delta(x_i - \overline{x}_n)\} > 0, \tag{2.73}$$

this is because $\Delta < 1$ and $(x_i - \overline{x}_l) \geq (x_i - \overline{x}_n)$ for all $1 \leq i \leq n$ ,(since $\overline{x}_l \leq \overline{x}_n, l \leq n$). Thus by (2.72) and under $\overline{H}_\circ$ the process in (2.71) on average does not equal to zero and oscillates between the two parameters $\beta$ and $\gamma$. Hence, a test for $\overline{H}_\circ$ can be based on the process $M_n(.,.)$ of (2.71) .

Motivated by the above discussion we suggest the use of the following statistics:

$$T_{n1} = \sup_{0<u<v<1} M_n(u,v), \tag{2.74}$$

for testing against $H_1$ in (2.69) and,

$$T_{n2} = \sup_{0<u<v<1} \mid M_n(u,v) \mid, \tag{2.75}$$

or,

$$T_{n3} = \int_0^1 \int_0^v M_n^2(u,v)\mathrm{d}u\mathrm{d}v, \tag{2.76}$$

for testing against $H_2$ in (2.69). For the test $T_{n3}$ we have by (2.70),

$$\begin{aligned} T_{n3} &= \int_0^1 \int_0^v (M_n(v) - M_n(u))^2 \mathrm{d}u\mathrm{d}v \\ &= \int_0^1 M_n^2(u)\mathrm{d}u - (\int_0^1 M_n(u)\mathrm{d}u)^2. \end{aligned} \tag{2.77}$$

We also recall that,

$$M_n(u) = \widehat{M}(k_*) = v_{k_*}^2 v_n^{-1} D_n(k_*), \qquad 0 < u < 1, \tag{2.78}$$

where $D_n(1) = D_n(n) = 0, D_n(m) = \hat{\beta}_m - \hat{\beta}_n, \; 2 \le m \le n-1$ and

$k_* = k_n(u) = \max\{k : (v_k^2/v_n^2) \le u\}$, $0 < u < 1$, which implies that;

$$k_n(u) = j \qquad \text{whenever} \qquad v_j^2 \le u\, v_n^2 < v_{j+1}^2, \;\; 2 \le j \le n-1. \tag{2.79}$$

Let $I_j = [(v_j^2/v_n^2), (v_{j+1}^2/v_n^2))$, then by (2.78) and (2.79) we obtain

$$\begin{aligned} \int_0^1 M_n(u)\mathrm{d}u &= \sum_{j=2}^{n-1} \int_{I_j} \widehat{M}(j)\mathrm{d}u \\ &= \sum_{j=2}^{n-1} \widehat{M}(j)(v_{j+1}^2 - v_j^2)/v_n^2. \end{aligned} \tag{2.80}$$

Similarly,

$$\int_0^1 M_n^2(u)\mathrm{d}u = \sum_{j=2}^{n-1} \widehat{M}^2(j)(v_{j+1}^2 - v_j^2)/v_n^2. \tag{2.81}$$

By (2.77), (2.80) and (2.81) we have,

$$T_{n3} = \sum_{j=2}^{n-1} \widehat{M}^2(j)(v_{j+1}^2 - v_j^2)/v_n^2 - \{\sum_{j=2}^{n-1} \widehat{M}(j)(v_{j+1}^2 - v_j^2)/v_n^2\}^2. \tag{2.82}$$

**Corollary** (2.2)

For the statistics in (2.74)-(2.76) we have,

$$\begin{aligned} T_{n1} &\xrightarrow{\mathrm{d}} \sup_{0<u<v<1} (B(v) - B(u)) := T_1, \\ T_{n2} &\xrightarrow{\mathrm{d}} \sup_{0<u<v<1} \mid B(v) - B(u) \mid := T_2, \\ T_{n3} &\xrightarrow{\mathrm{d}} \int_0^1 \int_0^v (B(v) - B(u))^2 \mathrm{d}u\mathrm{d}v := T_3, \end{aligned}$$

where $B(.)$ is a standard Brownian bridge.

**Proof**

Using Theorem (3.2) of Sen (1980) and the continuous mapping Theorem we obtain,

$$M_n(u,v) \xrightarrow{d} (B(v) - B(u)), \qquad 0 < u < v < 1,$$

and hence the proof is complete by applying the suitable continuous functional for each statistic.

Note that $T_2$ is the limiting rv of Kuiper (1960)'s statistic and $T_3$ is the limiting rv of Watson (1961)'s statistic. The distributions of $T_2$ and $T_3$ are tabulated in Shorack and Wellner (1986). Some asymptotic quantiles of $T_{n1}$ will be simulated in section 5.

### 2.4.2 A test for the two change points alternative

Here we consider the problem of testing the null hypothesis of no change in the regression model (2.1) against the alternative hypothesis described by the model,

$$Y_i = \alpha + \begin{cases} \beta_1 x_i + \epsilon_i & \text{for } 1 \le i \le k \\ \beta_2 x_i + \epsilon_i & \text{for } k+1 \le i \le l \\ \beta_3 x_i + \epsilon_i & \text{for } l+1 \le i \le n \end{cases}, \tag{2.83}$$

where the parameters $\alpha$, $\beta_1$, $\beta_2$, $\beta_3$ and the change points $k$ and $l$ are unknown. This is equivalent to testing:

$$H_o : \beta_1 = \beta_2 = \beta_3 \quad \text{against} \quad H_1 : \beta_1 \neq \beta_2 \neq \beta_3. \tag{2.84}$$

In a similar fashion as in the epidemic alternative case, we define here the main process $\{\Lambda_n^2(u,v) : 0 < u < v < 1\}$ by:

$$\Lambda_n^2(u,v) = M_n^2(u) + (M_n(v) - M_n(u))^2 + M_n^2(v), \quad 0 < u < v < 1, \tag{2.85}$$

where $M_n(.)$ is defined by (2.11).

To test the hypothesis in (2.84) we propose the following statistic;

$$T_n^2 = \int_0^1 \int_0^v \Lambda_n^2(u,v)\mathrm{d}u\mathrm{d}v. \tag{2.86}$$

**Corollary** (2.3)

$$T_n^2 \xrightarrow{\mathrm{d}} \int_0^1 \int_0^v \{B^2(u) + (B(v) - B(u))^2 + B^2(v)\}\mathrm{d}u\mathrm{d}v,$$

where $B(.)$ is a standard Brownian bridge.

The proof of this Corollary is exactly as in Corollary (2.2). Note that, quantiles for the distribution of the above limiting rv are given in Lombard (1987). It is also easy to see that

$$\begin{aligned} T_n^2 &= 2\int_0^1 M_n^2(u)\mathrm{d}u - (\int_0^1 M_n(u)\mathrm{d}u)^2 \\ &= 2\sum_{j=2}^{n-1} \widehat{M}^2(j)(v_{j+1}^2 - v_j^2)/v_n^2 - \{\sum_{j=2}^{n-1} \widehat{M}(j)(v_{j+1}^2 - v_j^2)/v_n^2\}^2. \end{aligned}$$

## 2.5 Estimated critical values and power comparisons

In this section we calculate through simulations the critical values and powers for the test statistics of this chapter. First we simulate the critical values and the powers of four LS, AMOC test statistics. These tests are; Sen's sup-type test, Cramér-von Mises (C-V), Anderson-Darling (A-D) and Erdős-Darling (E-D) type tests. We recall that these statistics are designed to test for a possible (one) change in the slope of a simple regression model. Their Monte Carlo results generally supported our expectations in two ways. The first is that the Monte Carlo critical values of our proposed weighted-type tests are very close to their respective limiting values. The second is that, the power results showed that these tests are sensitive to changes which occur close to the tails of the data set. Second we estimated the critical values of the epidemic statistics in (2.74)-(2.76) and the critical values of the AMTC test in (2.86). We also approximated the quantiles of the statistic in (2.74). In all the cases the estimated critical values seem to converge to the corresponding asymptotic quantiles. Finally we estimated some power values for the epidemic and the AMTC test statistics.

### Estimated critical values

We first conducted a Monte Carlo simulation study to estimate the critical values of the four test statistics; Sen's test, and the proposed Cramér-von Mises, Anderson-Darling and Erdős-Darling type tests. In this study we used $5,000$ realizations of the simple regression model $Y_i = \beta(i/n) + \epsilon_i$, $i = 1, \ldots, n$. These simulations were done for different sample sizes and three distributions for the regression error term, namely Normal, Exponential and Double-Exponential. In each simulation, sample size and distribution , the four test statistics were calculated. We ordered the $5,000$ values of each test and obtained the 95% percentiles of these values. The resulting critical values are give in Table 1. The estimated critical values, are generally close to the corresponding theoretical ones. But we can see also that it converge faster in the following order: Sen, Cramér-von Mises, Anderson-Darling and Erdős-Darling. Following the above technique we also estimated the critical values of the epidemic-type tests $T_{n1}, T_{n2}, T_{n3}$ of (2.74)-(2.76) and the AMTC test $T_n^2$ of (2.86). As above these estimated critical values were obtained for three different regression error distributions, namely Normal, Exponential and Double exponential and for sample sizes $n = 10, 20, \ldots, 100$. The results of these estimated critical values are shown in Tables 8-11. Furthermore, since the quantiles of the asymptotic distribution of $T_{n1}$ are not available, we calculated them through a Monte Carlo simulation. We generated a vector $Z = (Z_1, \ldots, Z_M)$ of multivariate Normal variates, with mean zero and covariance function equal to $t_i(1 - t_j)$, $1 \leq i \leq j \leq M$, where $t_i = \frac{i}{M+1}$, $i = 1, \ldots, M$. This vector is a discrete trajectory "in distribution" version of the Brownian bridge process $B(t)$ at $t = t_i$, $i = 1, \ldots, M$. We took $M = 800$ and used 1,000 realizations. In each realization we calculated the quantity $\max_{1 \leq i < j \leq M}(Z_j - Z_i)$ as an estimate of $\sup_{0 \leq u < v \leq 1}(B(v) - B(u))$. We then ordered the 1,000 values and obtain their $(1 - \alpha)^{th}$ percentiles for $\alpha = 0.1, 0.05$ and 0.01. These approximated quantiles together with the quantiles of the asymptotic distributions of $T_{n2}$, $T_{n3}$ and $T_n^2$ that we found in the literature are presented in the last row of Tables 8-11. The values of Tables 8-11 show that the overall agreement between the estimated critical values and the corresponding asymptotic quantiles is satisfactory. If anything, use of the approximated quantiles in small samples seem to give slightly conservative (liberal) tests in $T_{n1}$ and $T_{n2}$ (in $T_{n3}$ and $T_n^2$), but as the sample size increases this problem disappears.

### Monte Carlo power comparisons

To study the performance of the one change point tests we estimated their powers as follows. We considered two sample sizes $n = 30$ and $n = 50$ and several alternatives to $H_\circ : \delta = 0$ (i.e., there is no change in the slope of the regression model) of the form $(k, \delta)$ , where $k$ is the change position and $\delta$ is the size of the change. For the distribution of the regression error we used the Normal, the Exponential and the Double-Exponential distributions. For each of these distributions we employed $k = 2, \ldots, 29$ when $n = 30$ and selected values of $k$ when $n = 50$. The amount of change $\delta$ corresponding to each value of $k$ was obtained by solving the equation $P(Y_{k+1} > Y_k) = 0.95$. For each power calculation, $3,000$ samples of size $n$ were generated under the alternative segmented model. We calculated the four tests in each of the $3,000$ samples and computed the fraction of times that each test value exceeded the corresponding empirical critical value. The resulting powers are shown in Tables 2-7. These results show that the Weighted tests (that is, the Anderson-Darling and Erdős-Darling type tests) have higher powers compared to Sen's test and the Cramér-von Mises type test when the change occur near the tails of the data set. The powers of Sen's test and the Cramér-von Mises type test are higher than those of the weighted tests when the change occurs near the middle. We notice that what we call the middle of the data is shifted to almost 3/4 of the data. This shifting of the middle is exactly what Jandhyla and MacNeill (1991) referred to as a result of the discontinuity of the used model. We also notice that the power values under the Double-Exponential distribution are higher than under the other distributions which may be the result of its long tail that produces amount of change larger than the other two distributions at the same probability.

To compare the epidemic test statistics, we conducted a simple Monte Carlo power study. In this study we set the sample size $n$ to 40. We considered different change positions $(k, l)$ and two change sizes $\delta = 0.5$ and $\delta = 2.0$. For each $(k, l)$ and $\delta$ we calculated the epidemic tests 3,000 times. We then obtained for each test the percentage of times that the test value exceeded the corresponding estimated critical value at $\alpha = 0.05$. The estimated powers were calculated for three different regression error distributions, namely Normal, Exponential and Double-Exponential. The resulting power estimates are shown in Tables 12-14. From these tables we can see that for the same change size, the powers increase when the change positions are close to the middle of the data set. This is because in these cases there is sufficient number of data with different models, which make the change detection easier. We also notice that generally $T_{n1}$ which is one-sided test has the highest powers and $T_{n3}$ is performing at least as $T_{n2}$.

Finally, we estimated some power values for the AMTC test of (2.86). We obtained these powers for sample sizes $n = 40, 60, 80$ and 100. We considered different change positions $(k, l)$ and different slope parameters. For each change position and set of parameters, we calculated the AMTC test $3,000$ times under the alternative hypothesis in (2.84). At $\alpha = 0.05$, we obtained the percentages of times that the test value exceeded the corresponding critical value. The powers were calculated using Normal, Exponential and Double-Exponential regression error distributions. The estimated power percentages are shown in Tables 15-26. From these tables we can see that the powers increase (decrease) when the changes occur near the middle (ends) of the data set. We also notice that the powers are generally low when the sample size is small (e.g., $n = 40$). As the sample size increases the estimated powers increase and the difference between powers under the different distributions diminishes.

## Table 1

Estimated critical values for the AMOC tests

at $\alpha = 0.05$

| *Dist.* of $\epsilon$ | *Test* | $n$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 10 | 20 | 30 | 40 | 50 | 100 | 200 | 1000 | $\infty$ |
| *Nor.* | *Sen* | 1.147 | 1.155 | 1.229 | 1.230 | 1.269 | 1.280 | 1.315 | 1.361 | 1.360 |
| | *C-V* | 0.567 | 0.463 | 0.486 | 0.474 | 0.479 | 0.479 | 0.457 | 0.461 | 0.461 |
| | *A-D* | 2.818 | 2.468 | 2.585 | 2.560 | 2.520 | 2.574 | 2.439 | 2.511 | 2.492 |
| | *E-D* | 2.517 | 2.534 | 2.680 | 2.668 | 2.683 | 2.867 | 2.944 | 3.113 | 3.663 |
| *Exp.* | *Sen* | 1.240 | 1.271 | 1.283 | 1.309 | 1.309 | 1.303 | 1.311 | 1.359 | 1.360 |
| | *C-V* | 0.609 | 0.531 | 0.509 | 0.523 | 0.513 | 0.474 | 0.457 | 0.481 | 0.461 |
| | *A-D* | 3.178 | 2.846 | 2.743 | 2.795 | 2.774 | 2.537 | 2.486 | 2.549 | 2.492 |
| | *E-D* | 3.099 | 3.426 | 3.576 | 3.583 | 3.696 | 3.751 | 3.684 | 3.941 | 3.663 |
| *D-E* | *Sen* | 1.181 | 1.222 | 1.251 | 1.252 | 1.248 | 1.274 | 1.302 | 1.348 | 1.360 |
| | *C-V* | 0.568 | 0.528 | 0.499 | 0.482 | 0.448 | 0.471 | 0.462 | 0.472 | 0.461 |
| | *A-D* | 2.952 | 2.813 | 2.681 | 2.602 | 2.598 | 2.526 | 2.482 | 2.558 | 2.492 |
| | *E-D* | 2.812 | 3.092 | 3.221 | 3.221 | 3.278 | 3.362 | 3.386 | 3.553 | 3.663 |

Table 2

Estimated power percentages for the AMOC tests

at $\alpha = 0.05$, n=30

(Normal error distribution)

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 3.8 | 8.6 | 9.7 | 3.2 |
| 3 | 4.9 | 11.4 | 12.9 | 3.7 |
| 4 | 5.8 | 13.7 | 15.3 | 4.1 |
| 5 | 6.8 | 15.2 | 16.5 | 4.6 |
| 6 | 7.2 | 15.8 | 16.7 | 5.1 |
| 7 | 7.0 | 15.4 | 15.8 | 6.1 |
| 8 | 6.6 | 14.1 | 14.5 | 7.8 |
| 9 | 5.6 | 12.3 | 12.8 | 10.2 |
| 10 | 4.7 | 10.1 | 11.2 | 13.6 |
| 11 | 3.6 | 8.6 | 11.3 | 18.0 |
| 12 | 3.2 | 7.5 | 13.9 | 24.0 |
| 13 | 4.7 | 9.8 | 20.5 | 30.3 |
| 14 | 12.0 | 15.0 | 30.8 | 36.3 |
| 15 | 26.1 | 25.0 | 42.7 | 43.2 |
| 16 | 43.8 | 39.7 | 56.7 | 50.1 |
| 17 | 59.1 | 56.7 | 68.5 | 56.0 |
| 18 | 71.9 | 70.9 | 76.7 | 61.5 |
| 19 | 80.5 | 81.5 | 83.9 | 65.7 |
| 20 | 86.2 | 88.2 | 88.2 | 68.2 |
| 21 | 89.2 | 91.8 | 91.1 | 71.2 |
| 22 | 91.5 | 93.8 | 92.6 | 71.1 |
| 23 | 92.0 | 94.6 | 93.1 | 71.1 |
| 24 | 91.7 | 94.7 | 93.2 | 69.4 |
| 25 | 89.2 | 93.3 | 91.9 | 65.5 |
| 26 | 82.9 | 89.9 | 88.0 | 57.2 |
| 27 | 69.1 | 79.8 | 79.9 | 44.9 |
| 28 | 38.8 | 56.2 | 60.5 | 27.1 |
| 29 | 8.6 | 20.9 | 27.3 | 9.3 |

Table 3

Estimated power percentages for the AMOC tests

at $\alpha = 0.05$, n=30

(Exponential error distribution)

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 5.9 | 9.6 | 10.5 | 10.6 |
| 3 | 7.1 | 11.9 | 13.2 | 10.7 |
| 4 | 8.4 | 13.7 | 15.4 | 10.5 |
| 5 | 9.2 | 15.1 | 16.5 | 10.5 |
| 6 | 9.1 | 15.0 | 15.8 | 9.4 |
| 7 | 9.0 | 15.6 | 15.9 | 9.2 |
| 8 | 8.5 | 14.1 | 14.2 | 9.3 |
| 9 | 7.4 | 12.4 | 12.6 | 9.9 |
| 10 | 6.3 | 10.9 | 11.9 | 11.7 |
| 11 | 5.1 | 9.5 | 11.9 | 15.0 |
| 12 | 4.2 | 9.0 | 14.6 | 19.2 |
| 13 | 4.9 | 10.1 | 18.5 | 26.2 |
| 14 | 8.8 | 14.3 | 26.5 | 32.3 |
| 15 | 20.3 | 22.2 | 39.0 | 39.8 |
| 16 | 36.9 | 37.0 | 51.4 | 47.4 |
| 17 | 54.1 | 53.1 | 63.7 | 53.6 |
| 18 | 68.7 | 67.7 | 73.2 | 58.7 |
| 19 | 78.8 | 78.8 | 84.1 | 63.8 |
| 20 | 85.6 | 86.7 | 86.5 | 67.4 |
| 21 | 89.4 | 90.9 | 89.4 | 69.8 |
| 22 | 91.2 | 93.7 | 91.6 | 71.3 |
| 23 | 92.3 | 94.9 | 93.0 | 70.3 |
| 24 | 92.6 | 94.9 | 93.1 | 68.1 |
| 25 | 90.6 | 93.8 | 92.1 | 63.1 |
| 26 | 84.2 | 90.0 | 87.7 | 54.0 |
| 27 | 66.2 | 77.8 | 76.7 | 40.1 |
| 28 | 34.4 | 51.4 | 55.1 | 25.5 |
| 29 | 9.6 | 19.6 | 24.2 | 13.1 |

Table 4

Estimated power percentages for the AMOC tests

at $\alpha = 0.05$, n=30

(Double-Exponential error distribution)

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 5.6 | 12.7 | 14.9 | 9.1 |
| 3 | 8.5 | 17.9 | 21.2 | 11.4 |
| 4 | 10.7 | 22.3 | 25.9 | 13.2 |
| 5 | 12.3 | 25.8 | 29.2 | 14.0 |
| 6 | 13.1 | 26.9 | 29.9 | 16.1 |
| 7 | 12,7 | 26.2 | 27.5 | 18.7 |
| 8 | 11.4 | 23.2 | 24.2 | 23.9 |
| 9 | 9.6 | 19.1 | 21.0 | 32.0 |
| 10 | 7.7 | 15.6 | 18.7 | 41.8 |
| 11 | 6.2 | 12.0 | 19.7 | 53.0 |
| 12 | 8.8 | 10.4 | 27.2 | 64.0 |
| 13 | 22.2 | 14.5 | 42.1 | 73.9 |
| 14 | 46.8 | 28.2 | 61.3 | 81.8 |
| 15 | 71.9 | 50.5 | 77.2 | 87.6 |
| 16 | 87.4 | 72.6 | 88.3 | 91.3 |
| 17 | 95.0 | 95.1 | 94.8 | 94.6 |
| 18 | 97.7 | 95.4 | 97.6 | 96.4 |
| 19 | 99.2 | 98.6 | 99.2 | 97.4 |
| 20 | 99.6 | 99.5 | 99.6 | 98.0 |
| 21 | 99.8 | 99.8 | 99.6 | 98.4 |
| 22 | 99.9 | 99.9 | 99.9 | 98.6 |
| 23 | 100.0 | 100.0 | 99.9 | 98.6 |
| 24 | 99.9 | 100.0 | 99.9 | 98.4 |
| 25 | 99.7 | 99.8 | 99.7 | 97.3 |
| 26 | 99.1 | 99.5 | 99.3 | 94.9 |
| 27 | 96.2 | 98.0 | 97.8 | 89.5 |
| 28 | 78.9 | 86.8 | 90.1 | 71.6 |
| 29 | 16.4 | 36.9 | 50.0 | 28.0 |

Table 5

Estimated power percentages

(Normal error distribution)

$\alpha = 0.05$, n=50

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 3.7 | 8.1 | 9.0 | 3.4 |
| 3 | 5.1 | 10.2 | 12.2 | 4.9 |
| 4 | 6.7 | 13.0 | 15.7 | 6.5 |
| 5 | 8.6 | 15.8 | 18.6 | 8.1 |
| 10 | 13.3 | 23.4 | 25.0 | 14.5 |
| 15 | 9.6 | 17.0 | 18.0 | 29.8 |
| 20 | 7.0 | 9.8 | 23.8 | 56.5 |
| 25 | 61.4 | 42.6 | 69.6 | 79.5 |
| 30 | 95.5 | 92.0 | 94.9 | 91.7 |
| 35 | 99.4 | 99.2 | 99.1 | 95.7 |
| 40 | 99.6 | 99.4 | 99.3 | 95.2 |
| 45 | 92.9 | 94.7 | 94.5 | 80.8 |
| 46 | 83.5 | 88.1 | 89.4 | 70.0 |
| 47 | 62.0 | 70.8 | 77.0 | 53.9 |
| 48 | 28.8 | 42.1 | 53.2 | 31.9 |
| 49 | 12.8 | 23.2 | 31.2 | 14.9 |

Table 6

Estimated power percentages

(Exponential error distribution)

$\alpha = 0.05$, n=50

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 5.7 | 7.9 | 8.8 | 12.8 |
| 3 | 6.7 | 9.7 | 11.0 | 14.9 |
| 4 | 8.1 | 11.8 | 14.1 | 15.9 |
| 5 | 9.6 | 14.0 | 17.4 | 16.1 |
| 10 | 13.9 | 21.4 | 23.2 | 15.1 |
| 15 | 9.8 | 15.1 | 16.9 | 26.7 |
| 20 | 6.4 | 10.0 | 23.3 | 55.1 |
| 25 | 63.2 | 43.0 | 68.7 | 82.5 |
| 30 | 95.4 | 91.8 | 94.9 | 93.1 |
| 35 | 99.2 | 99.0 | 98.9 | 96.4 |
| 40 | 99.6 | 99.7 | 99.5 | 95.7 |
| 45 | 94.7 | 96.1 | 95.9 | 80.9 |
| 46 | 85.6 | 89.3 | 90.6 | 70.1 |
| 47 | 59.2 | 70.5 | 76.3 | 50.4 |
| 48 | 27.5 | 41.0 | 49.5 | 31.1 |
| 49 | 8.9 | 15.7 | 21.5 | 15.2 |

Table 7

Estimated power percentages

(Double-Exponential error distribution)

$\alpha = 0.05$, n=50

| *k* | *Test* | | | |
|---|---|---|---|---|
| | *Sen* | *C-V* | *A-D* | *E-D* |
| 2 | 4.3 | 10.0 | 11.8 | 11.8 |
| 3 | 6.3 | 14.9 | 17.9 | 15.8 |
| 4 | 9.8 | 19.6 | 24.8 | 18.9 |
| 5 | 14.0 | 25.1 | 31.2 | 23.1 |
| 10 | 24.2 | 38.3 | 42.6 | 38.8 |
| 15 | 16.5 | 27.0 | 32.2 | 73.2 |
| 20 | 36.4 | 37.1 | 56.5 | 94.9 |
| 25 | 97.8 | 82.9 | 96.7 | 99.4 |
| 30 | 100.0 | 99.9 | 100.0 | 99.9 |
| 35 | 100.0 | 100.0 | 100.0 | 100.0 |
| 40 | 100.0 | 100.0 | 100.0 | 100.0 |
| 45 | 99.9 | 99.9 | 99.9 | 99.8 |
| 46 | 99.6 | 99.7 | 99.7 | 98.9 |
| 47 | 96.0 | 96.6 | 98.4 | 95.3 |
| 48 | 66.1 | 75.9 | 87.8 | 81.9 |
| 49 | 11.9 | 25.7 | 39.3 | 30.3 |

Table 8

Estimated critical values for the epidemic-type tests

(Normal error distribution)

| $n$ | $T_{n1}$ | | | $T_{n2}$ | | | $T_{n3}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | | | $\alpha$ | | | $\alpha$ | |
| | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 |
| 10 | 1.04 | 1.18 | 1.47 | 1.16 | 1.28 | 1.57 | 0.159 | 0.200 | 0.294 |
| 20 | 1.17 | 1.30 | 1.58 | 1.28 | 1.42 | 1.70 | 0.154 | 0.189 | 0.265 |
| 30 | 1.24 | 1.38 | 1.62 | 1.35 | 1.47 | 1.72 | 0.151 | 0.188 | 0.284 |
| 40 | 1.27 | 1.41 | 1.65 | 1.39 | 1.53 | 1.77 | 0.156 | 0.190 | 0.278 |
| 50 | 1.29 | 1.44 | 1.74 | 1.41 | 1.53 | 1.80 | 0.156 | 0.189 | 0.278 |
| 60 | 1.34 | 1.47 | 1.71 | 1.43 | 1.57 | 1.81 | 0.151 | 0.187 | 0.271 |
| 70 | 1.33 | 1.46 | 1.74 | 1.43 | 1.57 | 1.83 | 0.152 | 0.185 | 0.272 |
| 80 | 1.35 | 1.48 | 1.78 | 1.46 | 1.60 | 1.83 | 0.151 | 0.186 | 0.273 |
| 90 | 1.35 | 1.48 | 1.75 | 1.45 | 1.57 | 1.83 | 0.152 | 0.187 | 0.273 |
| 100 | 1.38 | 1.52 | 1.78 | 1.47 | 1.60 | 1.84 | 0.153 | 0.188 | 0.277 |
| $\infty$ | 1.48 | 1.63 | 1.89 | 1.61 | 1.74 | 2.00 | 0.152 | 0.187 | 0.269 |

Table 9

Estimated critical values for the epidemic-type tests

(Exponential error distribution)

| $n$ | $T_{n1}$ | | | $T_{n2}$ | | | $T_{n3}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | | | $\alpha$ | | | $\alpha$ | |
| | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 |
| 10 | 1.14 | 1.35 | 1.92 | 1.26 | 1.50 | 1.99 | 0.180 | 0.255 | 0.475 |
| 20 | 1.26 | 1.48 | 1.89 | 1.39 | 1.61 | 1.99 | 0.170 | 0.225 | 0.391 |
| 30 | 1.31 | 1.46 | 1.86 | 1.41 | 1.58 | 1.95 | 0.162 | 0.222 | 0.357 |
| 40 | 1.34 | 1.52 | 1.92 | 1.45 | 1.63 | 2.01 | 0.168 | 0.219 | 0.339 |
| 50 | 1.34 | 1.49 | 1.84 | 1.43 | 1.59 | 1.93 | 0.162 | 0.208 | 0.308 |
| 60 | 1.37 | 1.53 | 1.88 | 1.46 | 1.63 | 2.01 | 0.159 | 0.208 | 0.336 |
| 70 | 1.39 | 1.54 | 1.85 | 1.48 | 1.62 | 1.95 | 0.160 | 0.207 | 0.326 |
| 80 | 1.39 | 1.56 | 1.87 | 1.49 | 1.66 | 1.94 | 0.164 | 0.202 | 0.317 |
| 90 | 1.41 | 1.57 | 1.88 | 1.50 | 1.65 | 1.94 | 0.159 | 0.197 | 0.293 |
| 100 | 1.41 | 1.57 | 1.89 | 1.50 | 1.65 | 1.96 | 0.156 | 0.198 | 0.284 |
| $\infty$ | 1.48 | 1.63 | 1.89 | 1.61 | 1.74 | 2.00 | 0.152 | 0.187 | 0.269 |

Table 10

Estimated critical values for the epidemic-type tests

(Double-Exponential error distribution)

| $n$ | $T_{n1}$ | | | $T_{n2}$ | | | $T_{n3}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | | | $\alpha$ | | | $\alpha$ | | |
| | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 |
| 10 | 1.03 | 1.23 | 1.65 | 1.19 | 1.36 | 1.81 | 0.172 | 0.236 | 0.405 |
| 20 | 1.20 | 1.38 | 1.73 | 1.33 | 1.49 | 1.82 | 0.170 | 0.217 | 0.326 |
| 30 | 1.26 | 1.41 | 1.72 | 1.37 | 1.53 | 1.82 | 0.165 | 0.209 | 0.319 |
| 40 | 1.28 | 1.43 | 1.75 | 1.39 | 1.53 | 1.88 | 0.157 | 0.201 | 0.299 |
| 50 | 1.30 | 1.46 | 1.75 | 1.41 | 1.55 | 1.86 | 0.159 | 0.198 | 0.301 |
| 60 | 1.31 | 1.46 | 1.76 | 1.43 | 1.56 | 1.83 | 0.156 | 0.197 | 0.288 |
| 70 | 1.35 | 1.47 | 1.76 | 1.44 | 1.56 | 1.81 | 0.157 | 0.194 | 0.296 |
| 80 | 1.33 | 1.48 | 1.80 | 1.44 | 1.59 | 1.89 | 0.157 | 0.201 | 0.304 |
| 90 | 1.35 | 1.48 | 1.75 | 1.46 | 1.58 | 1.87 | 0.156 | 0.197 | 0.294 |
| 100 | 1.36 | 1.50 | 1.76 | 1.46 | 1.60 | 1.85 | 0.154 | 0.195 | 0.287 |
| $\infty$ | 1.48 | 1.63 | 1.89 | 1.61 | 1.74 | 2.00 | 0.152 | 0.187 | 0.269 |

Table 11

Estimated critical values for the AMTC test $T_n^2$

| $n$ | $\epsilon \sim$*Normal* | | | $\epsilon \sim$*Exponential* | | | $\epsilon \sim$*D-Exponential* | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\alpha$ | | | $\alpha$ | | | $\alpha$ | | |
| | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 | 0.1 | 0.05 | 0.01 |
| 10 | 0.542 | 0.685 | 1.035 | 0.617 | 0.889 | 1.559 | 0.582 | 0.791 | 1.415 |
| 20 | 0.505 | 0.661 | 0.998 | 0.573 | 0.791 | 1.331 | 0.554 | 0.743 | 1.183 |
| 30 | 0.495 | 0.636 | 0.994 | 0.523 | 0.685 | 1.183 | 0.514 | 0.713 | 1.132 |
| 40 | 0.492 | 0.636 | 1.004 | 0.538 | 0.704 | 1.151 | 0.514 | 0.673 | 1.102 |
| 50 | 0.489 | 0.640 | 0.989 | 0.504 | 0.663 | 1.102 | 0.513 | 0.654 | 1.072 |
| 60 | 0.501 | 0.636 | 0.980 | 0.505 | 0.672 | 1.061 | 0.513 | 0.664 | 1.052 |
| 70 | 0.490 | 0.629 | 1.002 | 0.512 | 0.663 | 1.059 | 0.492 | 0.653 | 0.971 |
| 80 | 0.482 | 0.628 | 0.971 | 0.509 | 0.676 | 1.035 | 0.492 | 0.661 | 1.010 |
| 90 | 0.482 | 0.615 | 0.924 | 0.505 | 0.662 | 1.089 | 0.491 | 0.631 | 0.954 |
| 100 | 0.496 | 0.639 | 1.005 | 0.518 | 0.674 | 1.024 | 0.487 | 0.625 | 0.968 |
| $\infty$ | 0.481 | 0.628 | 0.956 | 0.481 | 0.628 | 0.956 | 0.481 | 0.628 | 0.956 |

Table 12

Estimated power percentages for the epidemic tests

at $\alpha = 0.05$ and $n = 40$.

( Normal regression errors)

|  |  | $\delta = 0.5$ |  |  | $\delta = 2.0$ |  |  |
|---|---|---|---|---|---|---|---|
| $k$ | $l$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ |
| 2 | 4 | 5.1 | 5.0 | 5.2 | 5.3 | 5.0 | 6.8 |
|  | 10 | 5.0 | 5.0 | 5.0 | 5.8 | 5.6 | 8.1 |
|  | 20 | 5.1 | 5.2 | 5.6 | 6.8 | 18.9 | 22.4 |
|  | 30 | 8.8 | 7.8 | 8.0 | 92.8 | 90.4 | 90.7 |
|  | 35 | 12.4 | 7.9 | 8.1 | 94.4 | 90.4 | 90.5 |
|  | 37 | 10.3 | 6.4 | 6.5 | 81.4 | 71.8 | 71.9 |
| 4 | 10 | 5.2 | 4.8 | 5.0 | 5.0 | 5.2 | 7.5 |
|  | 20 | 5.0 | 4.9 | 5.0 | 8.2 | 20.7 | 23.7 |
|  | 30 | 9.1 | 7.8 | 7.9 | 94.0 | 92.1 | 92.5 |
|  | 35 | 12.2 | 7.5 | 8.3 | 94.4 | 90.4 | 90.6 |
|  | 37 | 9.9 | 6.6 | 6.7 | 80.9 | 71.2 | 72.8 |
| 10 | 20 | 5.0 | 5.1 | 5.3 | 13.3 | 22.0 | 25.7 |
|  | 30 | 10.9 | 8.7 | 9.1 | 95.8 | 94.2 | 94.6 |
|  | 35 | 13.4 | 8.6 | 8.9 | 96.5 | 92.9 | 93.4 |
|  | 37 | 10.5 | 6.4 | 6.6 | 84.8 | 77.1 | 78.1 |
| 20 | 30 | 11.6 | 8.3 | 9.0 | 95.8 | 92.8 | 94.2 |
|  | 35 | 15.8 | 10.5 | 11.1 | 97.9 | 96.6 | 97.2 |
|  | 37 | 12.8 | 8.6 | 8.7 | 94.9 | 91.5 | 93.7 |
| 30 | 35 | 10.2 | 6.8 | 7.2 | 86.9 | 80.3 | 84.8 |
|  | 37 | 11.5 | 8.4 | 9.7 | 95.6 | 91.8 | 94.9 |
| 35 | 37 | 6.4 | 6.0 | 6.1 | 34.1 | 28.9 | 32.1 |

## Table 13

Estimated power percentages for the epidemic tests

at $\alpha = 0.05$ and $n = 40$.

( Exponential regression errors)

| | | $\delta = 0.5$ | | | $\delta = 2.0$ | | |
|---|---|---|---|---|---|---|---|
| $k$ | $l$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ |
| 2 | 4 | 5.1 | 4.9 | 5.0 | 4.9 | 4.8 | 6.1 |
| | 10 | 5.0 | 5.1 | 5.0 | 5.0 | 5.0 | 5.5 |
| | 20 | 5.2 | 5.4 | 5.3 | 7.8 | 14.2 | 15.5 |
| | 30 | 7.9 | 6.7 | 7.1 | 88.7 | 86.9 | 87.1 |
| | 35 | 6.8 | 5.4 | 6.5 | 88.1 | 83.7 | 85.0 |
| | 37 | 6.8 | 5.6 | 5.8 | 66.5 | 54.9 | 59.6 |
| 4 | 10 | 4.9 | 5.0 | 4.9 | 5.0 | 5.1 | 6.1 |
| | 20 | 5.1 | 4.9 | 5.0 | 7.6 | 14.9 | 17.0 |
| | 30 | 8.2 | 7.1 | 7.3 | 89.4 | 88.4 | 88.5 |
| | 35 | 8.4 | 6.8 | 7.3 | 89.5 | 86.5 | 87.6 |
| | 37 | 6.8 | 5.2 | 5.5 | 66.6 | 55.5 | 59.4 |
| 10 | 20 | 5.0 | 5.6 | 5.3 | 11.6 | 17.8 | 17.7 |
| | 30 | 7.7 | 7.0 | 7.2 | 94.2 | 93.3 | 92.6 |
| | 35 | 8.2 | 6.3 | 6.8 | 92.2 | 89.2 | 90.1 |
| | 37 | 7.4 | 5.1 | 5.6 | 73.9 | 65.3 | 69.3 |
| 20 | 30 | 8.7 | 7.0 | 7.1 | 92.1 | 89.2 | 89.9 |
| | 35 | 9.1 | 6.8 | 7.9 | 97.5 | 95.7 | 96.9 |
| | 37 | 8.4 | 5.8 | 6.9 | 90.5 | 85.7 | 88.9 |
| 30 | 35 | 6.6 | 5.1 | 6.3 | 82.6 | 73.1 | 77.9 |
| | 37 | 8.0 | 6.5 | 7.8 | 93.9 | 90.7 | 92.0 |
| 35 | 37 | 5.1 | 5.1 | 5.2 | 24.0 | 20.0 | 25.0 |

Table 14

Estimated power percentages for the epidemic tests

at $\alpha = 0.05$ and $n = 40$.

( Double-Exponential regression errors)

| | | $\delta = 0.5$ | | | $\delta = 2.0$ | | |
|---|---|---|---|---|---|---|---|
| $k$ | $l$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ | $T_{n1}$ | $T_{n2}$ | $T_{n3}$ |
| 2 | 4 | 5.4 | 5.8 | 6.4 | 5.9 | 6.0 | 6.5 |
| | 10 | 6.0 | 6.0 | 6.2 | 6.3 | 6.9 | 6.9 |
| | 20 | 5.0 | 6.9 | 7.0 | 8.9 | 21.4 | 21.3 |
| | 30 | 9.1 | 8.7 | 8.9 | 93.7 | 92.4 | 93.1 |
| | 35 | 12.1 | 9.4 | 9.6 | 94.4 | 91.6 | 93.2 |
| | 37 | 10.3 | 7.8 | 7.9 | 80.3 | 72.5 | 74.2 |
| 4 | 10 | 5.4 | 6.6 | 6.5 | 5.6 | 6.7 | 7.1 |
| | 20 | 5.8 | 7.2 | 7.4 | 10.4 | 21.8 | 21.9 |
| | 30 | 10.1 | 9.3 | 8.8 | 93.9 | 92.8 | 93.1 |
| | 35 | 11.3 | 8.9 | 9.1 | 94.6 | 92.2 | 93.2 |
| | 37 | 9.8 | 7.3 | 7.3 | 80.9 | 72.5 | 74.2 |
| 10 | 20 | 5.2 | 6.8 | 6.8 | 14.6 | 23.7 | 23.6 |
| | 30 | 11.6 | 10.3 | 10.6 | 96.8 | 95.5 | 95.7 |
| | 35 | 12.6 | 10.0 | 10.6 | 96.4 | 94.5 | 95.1 |
| | 37 | 11.2 | 8.4 | 8.9 | 86.2 | 79.0 | 82.1 |
| 20 | 30 | 11.5 | 10.3 | 10.4 | 95.2 | 93.1 | 94.1 |
| | 35 | 15.7 | 11.7 | 12.2 | 98.6 | 97.7 | 97.9 |
| | 37 | 13.8 | 9.9 | 10.1 | 95.6 | 93.0 | 94.1 |
| 30 | 35 | 10.2 | 8.5 | 8.6 | 87.5 | 82.5 | 83.1 |
| | 37 | 10.0 | 8.8 | 10.1 | 97.2 | 95.1 | 96.3 |
| 35 | 37 | 6.8 | 7.1 | 7.1 | 32.6 | 29.4 | 31.0 |

Table 15

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 40$.

( Normal regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 2 | 4 | 5.5 | 5.5 | 5.5 |
| | 10 | 5.5 | 5.1 | 5.5 |
| | 20 | 5.0 | 5.7 | 5.9 |
| | 30 | 10.6 | 27.8 | 28.4 |
| | 35 | 9.6 | 27.7 | 27.9 |
| | 37 | 7.5 | 17.3 | 17.9 |
| 4 | 10 | 5.5 | 5.9 | 6.0 |
| | 20 | 5.0 | 5.7 | 6.1 |
| | 30 | 9.4 | 26.4 | 27.0 |
| | 35 | 9.6 | 26.2 | 26.6 |
| | 37 | 7.6 | 17.9 | 18.4 |
| 10 | 20 | 5.7 | 5.7 | 5.8 |
| | 30 | 10.5 | 25.9 | 26.8 |
| | 35 | 10.2 | 26.0 | 26.9 |
| | 37 | 7.4 | 15.9 | 16.1 |
| 20 | 30 | 11.1 | 31.2 | 34.9 |
| | 35 | 10.6 | 28.7 | 30.9 |
| | 37 | 9.9 | 17.1 | 19.0 |
| 30 | 35 | 27.3 | 51.0 | 73.1 |
| | 37 | 19.6 | 37.4 | 60.4 |
| 35 | 37 | 21.5 | 39.8 | 68.4 |

Table 16

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 40$.

( Exponential regression errors)

| | | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 2 | 4 | 5.0 | 5.5 | 5.8 |
| | 10 | 4.8 | 4.9 | 5.1 |
| | 20 | 4.9 | 6.2 | 6.9 |
| | 30 | 9.3 | 23.0 | 23.1 |
| | 35 | 9.3 | 23.0 | 23.3 |
| | 37 | 7.0 | 13.7 | 14.7 |
| 4 | 10 | 5.3 | 5.2 | 5.3 |
| | 20 | 4.9 | 5.6 | 5.9 |
| | 30 | 8.1 | 21.0 | 21.2 |
| | 35 | 9.3 | 21.8 | 22.8 |
| | 37 | 7.1 | 13.5 | 13.9 |
| 10 | 20 | 4.8 | 4.9 | 5.5 |
| | 30 | 8.2 | 21.2 | 21.4 |
| | 35 | 8.3 | 20.3 | 20.4 |
| | 37 | 7.2 | 14.4 | 15.2 |
| 20 | 30 | 10.0 | 25.9 | 30.5 |
| | 35 | 10.5 | 23.6 | 25.4 |
| | 37 | 7.8 | 14.7 | 15.9 |
| 30 | 35 | 21.4 | 45.1 | 69.2 |
| | 37 | 16.4 | 30.2 | 53.9 |
| 35 | 37 | 18.4 | 35.6 | 61.6 |

Table 17

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 40$.

( Double-Exponential regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 2 | 4 | 5.7 | 5.1 | 5.1 |
| | 10 | 5.8 | 5.9 | 6.1 |
| | 20 | 5.2 | 5.7 | 5.6 |
| | 30 | 10.6 | 28.7 | 29.6 |
| | 35 | 10.5 | 29.0 | 30.2 |
| | 37 | 7.8 | 19.3 | 20.3 |
| 4 | 10 | 5.8 | 6.1 | 6.4 |
| | 20 | 5.0 | 6.6 | 6.7 |
| | 30 | 9.2 | 26.9 | 27.9 |
| | 35 | 9.4 | 26.5 | 27.5 |
| | 37 | 7.7 | 19.5 | 20.1 |
| 10 | 20 | 5.6 | 5.6 | 5.8 |
| | 30 | 10.4 | 26.1 | 27.0 |
| | 35 | 10.3 | 26.4 | 27.2 |
| | 37 | 7.4 | 16.3 | 18.2 |
| 20 | 30 | 11.5 | 31.5 | 35.6 |
| | 35 | 11.4 | 29.3 | 31.7 |
| | 37 | 10.6 | 20.3 | 22.6 |
| 30 | 35 | 28.0 | 51.1 | 74.0 |
| | 37 | 20.2 | 38.8 | 61.9 |
| 35 | 37 | 21.6 | 41.4 | 70.9 |

Table 18

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 60$.

( Normal regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 9 | 15 | 5.4 | 5.8 | 5.7 |
| | 21 | 5.5 | 6.0 | 5.9 |
| | 33 | 6.5 | 8.7 | 12.5 |
| | 45 | 11.4 | 35.4 | 70.0 |
| | 51 | 12.0 | 42.2 | 74.3 |
| | 57 | 8.2 | 18.4 | 42.8 |
| 15 | 21 | 5.2 | 6.1 | 5.9 |
| | 33 | 5.6 | 8.8 | 13.6 |
| | 45 | 11.3 | 35.3 | 65.0 |
| | 51 | 13.0 | 39.9 | 72.4 |
| | 57 | 8.6 | 20.5 | 40.0 |
| 21 | 33 | 6.3 | 8.4 | 14.4 |
| | 45 | 11.8 | 34.5 | 67.6 |
| | 51 | 12.8 | 40.8 | 73.3 |
| | 57 | 7.8 | 19.7 | 41.0 |
| 33 | 45 | 19.4 | 48.3 | 86.8 |
| | 51 | 19.0 | 49.4 | 85.6 |
| | 57 | 12.1 | 25.0 | 54.7 |
| 45 | 51 | 37.1 | 72.8 | 98.8 |
| | 57 | 26.0 | 46.6 | 89.4 |
| 51 | 57 | 31.0 | 52.3 | 94.2 |

### Table 19

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 60$.

( Exponential regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 9 | 15 | 5.0 | 5.2 | 5.4 |
| | 21 | 5.0 | 5.7 | 6.2 |
| | 33 | 5.2 | 8.1 | 11.8 |
| | 45 | 11.4 | 35.9 | 64.3 |
| | 51 | 11.6 | 39.9 | 72.5 |
| | 57 | 7.6 | 19.3 | 36.7 |
| 15 | 21 | 5.1 | 5.6 | 5.9 |
| | 33 | 5.6 | 8.5 | 12.2 |
| | 45 | 11.0 | 34.2 | 61.9 |
| | 51 | 11.6 | 39.5 | 68.2 |
| | 57 | 8.5 | 18.9 | 34.8 |
| 21 | 33 | 5.3 | 8.6 | 13.2 |
| | 45 | 9.9 | 35.4 | 62.8 |
| | 51 | 11.5 | 38.1 | 68.7 |
| | 57 | 7.8 | 19.4 | 35.8 |
| 33 | 45 | 17.3 | 48.9 | 84.9 |
| | 51 | 16.1 | 47.6 | 85.2 |
| | 57 | 10.6 | 23.5 | 50.9 |
| 45 | 51 | 34.6 | 72.0 | 98.7 |
| | 57 | 22.6 | 44.4 | 88.8 |
| 51 | 57 | 26.6 | 51.8 | 94.6 |

Table 20

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 60$.

( Double-Exponential regression errors)

| | | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 9 | 15 | 5.1 | 5.2 | 5.4 |
| | 21 | 5.3 | 6.1 | 5.5 |
| | 33 | 5.9 | 7.9 | 12.4 |
| | 45 | 11.3 | 34.9 | 66.3 |
| | 51 | 11.5 | 39.1 | 73.0 |
| | 57 | 7.5 | 18.4 | 40.8 |
| 15 | 21 | 5.1 | 5.6 | 6.0 |
| | 33 | 5.5 | 8.6 | 12.3 |
| | 45 | 11.5 | 34.9 | 66.0 |
| | 51 | 11.7 | 38.9 | 70.5 |
| | 57 | 8.4 | 18.2 | 36.4 |
| 21 | 33 | 5.7 | 8.3 | 13.9 |
| | 45 | 10.6 | 35.4 | 63.7 |
| | 51 | 12.2 | 38.1 | 69.8 |
| | 57 | 7.6 | 19.5 | 37.8 |
| 33 | 45 | 17.0 | 48.1 | 85.4 |
| | 51 | 16.9 | 47.9 | 84.3 |
| | 57 | 10.9 | 24.1 | 53.9 |
| 45 | 51 | 35.8 | 72.1 | 98.8 |
| | 57 | 23.4 | 44.5 | 89.7 |
| 51 | 57 | 26.4 | 51.8 | 94.0 |

Table 21

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 80$.

( Normal regression errors)

| | | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 12 | 20 | 5.1 | 5.5 | 6.2 |
| | 28 | 5.4 | 5.8 | 6.1 |
| | 44 | 5.6 | 10.5 | 17.6 |
| | 60 | 15.0 | 46.0 | 81.3 |
| | 68 | 16.1 | 50.3 | 84.9 |
| | 76 | 9.2 | 24.1 | 50.1 |
| 20 | 28 | 5.3 | 6.2 | 6.9 |
| | 44 | 5.4 | 10.0 | 14.7 |
| | 60 | 13.2 | 46.7 | 78.8 |
| | 68 | 14.3 | 50.6 | 83.1 |
| | 76 | 7.4 | 23.0 | 46.9 |
| 28 | 44 | 6.7 | 9.6 | 17.7 |
| | 60 | 14.5 | 45.4 | 79.5 |
| | 68 | 15.3 | 51.1 | 83.5 |
| | 76 | 8.3 | 23.5 | 47.1 |
| 44 | 60 | 23.5 | 60.2 | 93.6 |
| | 68 | 23.4 | 61.4 | 93.9 |
| | 76 | 12.4 | 30.5 | 64.8 |
| 60 | 68 | 49.0 | 83.9 | 99.8 |
| | 76 | 32.2 | 56.1 | 96.1 |
| 68 | 76 | 37.2 | 63.2 | 98.0 |

Table 22

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 80$.

( Exponential regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 12 | 20 | 5.2 | 5.7 | 5.6 |
| | 28 | 5.0 | 5.1 | 6.3 |
| | 44 | 5.1 | 9.4 | 16.3 |
| | 60 | 13.0 | 42.6 | 78.2 |
| | 68 | 13.8 | 45.3 | 79.9 |
| | 76 | 7.5 | 21.6 | 48.2 |
| 20 | 28 | 5.1 | 5.1 | 5.8 |
| | 44 | 6.3 | 9.0 | 13.3 |
| | 60 | 12.0 | 41.7 | 77.4 |
| | 68 | 13.2 | 45.6 | 79.8 |
| | 76 | 7.1 | 18.9 | 42.1 |
| 28 | 44 | 5.0 | 9.3 | 17.1 |
| | 60 | 12.8 | 40.6 | 78.0 |
| | 68 | 13.5 | 44.8 | 80.3 |
| | 76 | 8.2 | 19.5 | 44.2 |
| 44 | 60 | 20.0 | 53.9 | 89.9 |
| | 68 | 20.4 | 57.9 | 90.5 |
| | 76 | 11.2 | 26.1 | 60.8 |
| 60 | 68 | 44.5 | 82.1 | 98.9 |
| | 76 | 27.1 | 49.5 | 94.2 |
| 68 | 76 | 33.2 | 58.7 | 95.6 |

Table 23

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 80$.

( Double-Exponential regression errors)

|  |  | $(\beta_1, \beta_2, \beta_3)$ |  |  |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 12 | 20 | 5.1 | 5.3 | 5.7 |
|  | 28 | 5.2 | 5.5 | 6.3 |
|  | 44 | 5.4 | 9.0 | 16.4 |
|  | 60 | 13.0 | 44.7 | 78.9 |
|  | 68 | 14.4 | 47.9 | 83.4 |
|  | 76 | 7.8 | 21.3 | 45.5 |
| 20 | 28 | 5.4 | 5.3 | 6.5 |
|  | 44 | 5.5 | 8.9 | 15.8 |
|  | 60 | 12.5 | 41.0 | 76.1 |
|  | 68 | 13.9 | 47.1 | 83.6 |
|  | 76 | 8.0 | 20.6 | 44.0 |
| 28 | 44 | 5.3 | 9.0 | 17.0 |
|  | 60 | 11.6 | 42.8 | 79.1 |
|  | 68 | 15.2 | 47.3 | 83.9 |
|  | 76 | 8.2 | 20.4 | 44.8 |
| 44 | 60 | 22.3 | 58.1 | 93.7 |
|  | 68 | 20.5 | 59.3 | 93.5 |
|  | 76 | 11.8 | 27.8 | 61.8 |
| 60 | 68 | 45.9 | 82.5 | 99.8 |
|  | 76 | 27.1 | 52.1 | 94.7 |
| 68 | 76 | 31.7 | 60.7 | 97.9 |

Table 24

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 100$.

( Normal regression errors)

| $k$ | $l$ | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| | | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 15 | 25 | 5.0 | 5.0 | 6.0 |
| | 35 | 5.4 | 5.3 | 6.2 |
| | 55 | 5.8 | 10.5 | 20.3 |
| | 75 | 15.0 | 53.9 | 88.0 |
| | 85 | 18.8 | 60.9 | 92.2 |
| | 95 | 8.6 | 27.0 | 57.1 |
| 25 | 35 | 5.1 | 5.6 | 6.7 |
| | 55 | 5.6 | 11.3 | 18.0 |
| | 75 | 15.4 | 54.5 | 86.7 |
| | 85 | 17.8 | 59.9 | 90.8 |
| | 95 | 7.8 | 25.3 | 53.8 |
| 35 | 55 | 6.2 | 11.6 | 22.5 |
| | 75 | 16.8 | 53.7 | 87.1 |
| | 85 | 17.5 | 58.5 | 90.6 |
| | 95 | 9.2 | 25.4 | 52.2 |
| 55 | 75 | 27.0 | 68.8 | 97.4 |
| | 85 | 27.9 | 70.9 | 97.5 |
| | 95 | 14.0 | 35.3 | 72.8 |
| 75 | 85 | 55.9 | 90.6 | 100.0 |
| | 95 | 37.9 | 62.6 | 98.1 |
| 85 | 95 | 42.0 | 70.0 | 99.6 |

Table 25

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 100$.

( Exponential regression errors)

| | | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 15 | 25 | 5.1 | 5.0 | 5.9 |
| | 35 | 5.0 | 5.2 | 5.8 |
| | 55 | 5.5 | 9.4 | 18.6 |
| | 75 | 14.5 | 51.1 | 86.1 |
| | 85 | 15.7 | 55.1 | 90.5 |
| | 95 | 8.1 | 22.9 | 51.3 |
| 25 | 35 | 5.0 | 5.0 | 6.3 |
| | 55 | 5.1 | 9.1 | 18.0 |
| | 75 | 16.1 | 48.6 | 84.0 |
| | 85 | 15.2 | 53.4 | 89.0 |
| | 95 | 8.0 | 23.5 | 48.1 |
| 35 | 55 | 5.9 | 9.9 | 19.1 |
| | 75 | 14.3 | 49.7 | 85.5 |
| | 85 | 14.8 | 55.5 | 90.0 |
| | 95 | 7.5 | 21.8 | 47.5 |
| 55 | 75 | 25.5 | 65.6 | 97.4 |
| | 85 | 24.1 | 67.1 | 97.0 |
| | 95 | 11.8 | 31.9 | 69.3 |
| 75 | 85 | 52.2 | 88.8 | 99.9 |
| | 95 | 34.2 | 58.8 | 98.0 |
| 85 | 95 | 37.8 | 68.1 | 99.4 |

Table 26

Estimated power percentages for the AMTC test

at $\alpha = 0.05$ and $n = 100$.

( Double-Exponential regression errors)

| | | $(\beta_1, \beta_2, \beta_3)$ | | |
|---|---|---|---|---|
| $k$ | $l$ | (0.0 , 0.5 , 1.0) | (0.0 , 0.5 , 1.5) | (0.0 , 1.0 , 2.0) |
| 15 | 25 | 6.0 | 5.9 | 6.0 |
| | 35 | 5.2 | 5.3 | 6.3 |
| | 55 | 5.7 | 10.1 | 20.3 |
| | 75 | 15.2 | 53.3 | 87.9 |
| | 85 | 18.6 | 60.1 | 92.4 |
| | 95 | 8.7 | 26.9 | 58.0 |
| 25 | 35 | 5.0 | 5.5 | 6.9 |
| | 55 | 5.4 | 11.1 | 17.9 |
| | 75 | 15.2 | 53.8 | 86.6 |
| | 85 | 18.0 | 59.5 | 91.0 |
| | 95 | 7.9 | 25.1 | 53.6 |
| 35 | 55 | 6.0 | 11.3 | 22.2 |
| | 75 | 16.5 | 53.4 | 86.9 |
| | 85 | 17.5 | 58.9 | 90.5 |
| | 95 | 8.9 | 25.3 | 51.9 |
| 55 | 75 | 27.9 | 68.8 | 97.2 |
| | 85 | 28.0 | 71.0 | 97.1 |
| | 95 | 14.9 | 35.2 | 72.9 |
| 75 | 85 | 55.5 | 90.8 | 100.0 |
| | 95 | 36.6 | 61.9 | 98.1 |
| 85 | 95 | 41.2 | 70.5 | 99.4 |

## 2.6 Appendix

**A**: Let $W(.)$ be a standard Wiener process and $W_n(.)$ be as in (2.21), then

$$W_n(s) \overset{\mathrm{d}}{=} W(v_{[ns]}^2),$$

where $v_i$ is defined by (2.4).

**Proof** : Let $k = [ns] \leq l = [nt]$, $x_m^* = (x_m - \overline{x}_m)$, $d_m = (x_{m+1} - x_m)$ and

$$C = Cov(W_n(s), W_n(t)). \tag{2.87}$$

Then,

$$\begin{aligned}
C &= E\{x_k^* W(k) - \sum_{i=1}^{k-1} W(i)d_i\}\{x_l^* W(l) - \sum_{i=1}^{l-1} W(i)d_i\} \\
&= E\{x_k^* x_l^* W(k)W(l) - x_k^* \sum_{i=1}^{l-1} W(i)d_i W(k) - x_l^* \sum_{i=1}^{k-1} W(i)d_i W(l) \\
&\quad + \sum_{i=1}^{l-1} W(i)d_i \sum_{i=1}^{k-1} W(i)d_i\} \\
&= x_k^* x_l^* k - x_k^* \sum_{i=1}^{k-1} id_i - kx_k^* \sum_{i=k}^{l-1} d_i - x_l^* \sum_{i=1}^{k-1} id_i + \sum_{j=1}^{k-1}\sum_{i=1}^{l-1} d_j d_j E\{W(j)W(i)\} \\
&= x_k^* x_l^* k - (x_k^* + x_l^*) \sum_{i=1}^{k-1} id_i - kx_k^*(x_l^* - x_k^*) + \sum_{j=1}^{k-1}\sum_{i=1}^{j} id_i d_j + \sum_{j=1}^{k-1}\sum_{i=j+1}^{l-1} jd_j d_i \\
&= x_k^* x_l^* k - (x_k^* + x_l^*) \sum_{i=1}^{k-1} id_i - kx_k^*(x_l^* - x_k^*) + \sum_{i=1}^{k-1} id_i \sum_{j=i}^{k-1} d_j \\
&\quad + \sum_{j=1}^{k-1} jd_j(x_l - x_{j+1}) \\
&= x_k^* x_l^* k - (x_k^* + x_l^*) \sum_{i=1}^{k-1} id_i - kx_k^*(x_l^* - x_k^*) + \sum_{i=1}^{k-1} id_i(x_k - x_i) \\
&\quad + \sum_{j=1}^{k-1} jd_j(x_l - x_{j+1})
\end{aligned}$$

$$\begin{aligned} &= x_k^* x_l^* k - (x_k^* + x_l^*) \sum_{i=1}^{k-1} i d_i - k x_k^* (x_l^* - x_k^*) + (x_k + x_l) \sum_{i=1}^{k-1} i d_i \\ &\quad - \sum_{i=1}^{k-1} i (x_{i+1}^2 - x_i^2) \\ &= k(x_k - \overline{x}_k)(x_k - \overline{x}_l) + (\overline{x}_k + \overline{x}_l) \sum_{i=1}^{k-1} i d_i - \sum_{i=1}^{k-1} i (x_{i+1}^2 - x_i^2). \end{aligned} \tag{2.88}$$

Using Abel's summation we obtain,

$$\begin{aligned} \sum_{i=1}^{k-1} i d_i &= k(x_k - x_1) - \sum_{i=1}^{k-1} (x_{i+1} - x_1) \\ &= k(x_k - \overline{x}_k), \end{aligned} \tag{2.89}$$

and

$$\sum_{i=1}^{k-1} i (x_{i+1}^2 - x_i^2) = k x_k^2 - \sum_{i=1}^{k} x_i^2. \tag{2.90}$$

Substituting (2.89) and (2.90) into (2.88) we get,

$$\begin{aligned} C &= k(x_k - \overline{x}_k)(x_k - \overline{x}_l) + (\overline{x}_k + \overline{x}_l) k((x_k - \overline{x}_k) - k x_k^2 + \sum_{i=1}^{k} x_i^2 \\ &= \sum_{i=1}^{k} x_i^2 - k \overline{x}_k^2 = v_k^2. \end{aligned} \tag{2.91}$$

Since, $v_k^2$ is increasing in k we also have,

$$Cov(W(v_k^2), W(v_l^2)) = v_k^2, \qquad k < l. \tag{2.92}$$

By (2.91) and (2.92) we completed the required proof.

**B** : Let the time interval $x_n - x_1 < \infty$, then

$$\max_{1 \le k \le n} \mid x_k - \overline{x}_k \mid < \infty$$

**Proof :**

$$\begin{aligned}
\max_{1\le k\le n} \mid x_k - \overline{x}_k \mid &= \max_{1\le k\le n}\{\frac{1}{k} \mid \sum_{i=1}^{k}(x_i - x_k) \mid\} \\
&\le \max_{1\le k\le n}\{\frac{1}{k} \sum_{i=1}^{k} \mid x_i - x_k \mid\} \\
&\le \max_{1\le k\le n}\{\frac{1}{k} \sum_{i=1}^{k} \mid x_1 - x_n \mid\} \\
&= \mid x_1 - x_n \mid < \infty.
\end{aligned}$$

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# Chapter 3

# Bayesian tests for epidemic alternatives in regression models

## 3.1 Introduction

Bayes-type statistics were first introduced by Chernoff and Zacks (1964) to detect a change in the mean of a sequence of independent rv's taken from the Normal distribution. This method was adopted by Gardner (1969), Sen and Srivastava (1973) to derive tests of a change in the parameters of Normal and multivariate Normal observations. MacNeill (1974) obtained Bayes-type tests for a change in the mean of a sequence of Exponential rv's. Sen and Srivastava (1975) made a comparison study between Bayes-type and Likelihood ratio statistics for a change in the mean of a sequence of Normal rv's. They found that the Bayes-type statistics provide powerful tests when small changes occur. MacNeill (1978) proposed a test statistic for testing against a change in regression parameters at unknown time point analogous to that of Gardner (1969). Jandhyala and MacNeill (1987) obtained various one-sided and two-sided Bayes-type tests for changes in regression parameters. They pointed out that the complete distribution theory for the one-sided tests is available but even the asymptotic theory for the two-sided case is very complicated. Jandhyala and MacNeill (1989) obtained the asymptotic theory for some Bayes-type two-sided tests in the case of harmonic regression. Jandhyala and MacNeill (1991) computed the asymptotic quantiles of Bayes-type tests for a change in $\beta_o$ and $\beta_1$ in the case of polynomial regression. Jandhyala and Minogue (1993) introduced a numerical method for solving the stochastic integrals involved in computing the asymptotic quantiles of the above Bayes-type statistics.

In this Chapter we will consider the problem of testing against two-sided epidemic (square) alternatives. In section 2 we give some notations, assumptions and formulate the problem. In section 3 we derive a test statistic for the epidemic-type problem. The convergence results and asymptotic distributions are given in section 4. In section 5 we give some applications of the proposed test and present the results of several Monte Carlo studies.

### 3.2 Notation and Assumptions

Let $Y_1, \ldots, Y_n$ be a sequence of independent observations taken in a successive manner (e.g., over time). The null hypothesis considered here is given by :

$$H_o : \text{the } Y's \text{ obey the regression model } \mathbf{Y} = \mathbf{X}\beta + \varepsilon, \tag{3.1}$$

where $\mathbf{Y} = (Y_1, \ldots, Y_n)', \varepsilon = (\epsilon_1, \ldots, \epsilon_n)' \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$, $\beta = (\beta_0, \ldots, \beta_m)$ is the parameter vector and $\mathbf{X}$ is the design matrix given by:

$$\mathbf{X} = \begin{pmatrix} 1 & x_{11} & x_{12} & \ldots & \ldots & .. & x_{1m} \\ 1 & x_{21} & x_{22} & \ldots & \ldots & .. & x_{2m} \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 1 & x_{n1} & x_{n2} & \ldots & \ldots & .. & x_{nm} \end{pmatrix}.$$

Suppose now that we are interested in testing $H_\circ$ of (3.1) against the alternative that an epidemic-type change in the regression parameter vector $\beta = (\beta_0, \ldots, \beta_m)$ has taken place. The epidemic alternative means a change in the value of the parameter vector occurs at unknown point $k$, (the onset time) and continues to hold up to and including an unknown point $l$ ($> k$) after which the parameter vector changes back to its original value. To formulate this problem, let $\delta$ be the amount of change in each component of the parameter vector and recall that under the hypothesis of no change, the regression model is as given in (3.1). Thus the problem now is to test whether or not the model has changed to

$$\mathbf{Y} = \mathbf{X}\beta + \delta \sum_{j=0}^{m} \mathbf{X}_{k,j}^{l} + \varepsilon, \tag{3.2}$$

for some unknown $k$ and $l$ such that $0 \leq k < l \leq n-1$; where $\mathbf{X}_{k,j}^{l}$ is the $j^{th}$ column vector of the design matrix $\mathbf{X}$, keeping only the components starting from the $(k+1)^{th}$ row up to the $l^{th}$ row and replacing the rest of the components by zero, i.e.,

$$\mathbf{X}_{k,j}^{l} = (0, \ldots, 0, x_{k+1,j}, \ldots, x_{l,j}, 0, \ldots, 0)'. \tag{3.3}$$

The above problem can be equivalently expressed in terms of the following hypotheses:

$$H_\circ : \delta = 0 \qquad \text{vs} \qquad H_1 : \delta \neq 0. \tag{3.4}$$

### 3.3 A 'Bayes-type' test statistic

The Bayesian method introduced by Chernoff and Zacks (1964) requires that we specify a prior distribution for each of the unknown parameters $\beta$, $\delta$ and for the change points $k$ and $l$. We assume here that the unknown parameters $\beta, \delta$ and $(k, l)$ have the prior distributions $\mathcal{N}(\mathbf{0}, \tau^2\mathbf{I})$ , $\mathcal{N}(0, \theta^2)$ and $P(k, l)$, respectively. We also assume that $\beta, \delta$ and $\varepsilon$ are independent. The derived test statistic is obtained by letting $\tau^2 \rightarrow \infty$ and $\theta^2 \rightarrow 0$.

**Theorem** (3.1)

A 'Bayes-type' Likelihood ratio statistic for the hypotheses in (3.4) is given by :

$$T_n = \sum_{0 \leq k < l \leq n-1} P(k,l)\mathbf{Y}'\mathbf{R}(\sum_{j=0}^{m} \mathbf{X}_{k,j}^{l})(\sum_{j=0}^{m} \mathbf{X}_{k,j}^{l})'\mathbf{R}\mathbf{Y},$$

where $\mathbf{R} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\mathbf{X}_{k,j}^{l}$ is as in (3.3).

**Proof:**

Since under $H_\circ$ of (3.4) we have $(\mathbf{Y}|\beta) \sim \mathcal{N}(\mathbf{X}\beta, \sigma^2\mathbf{I})$ and the prior of $\beta$ is $\mathcal{N}(\mathbf{0}, \tau^2\mathbf{I})$, then by Lemma (2.1) of Jandhyala and MacNeill (1989), we have $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \Sigma_\circ)$, where $\Sigma_\circ = \sigma^2\mathbf{I} + \tau^2(\mathbf{X}'\mathbf{X})$. Thus the Bayesian Likelihood function under $H_\circ$ is given by

$$L_\circ(\mathbf{Y}) = (2\pi)^{-\frac{n}{2}} \mid \Sigma_\circ \mid^{-\frac{1}{2}} \exp(-\frac{1}{2}\mathbf{Y}'\Sigma_\circ^{-1}\mathbf{Y}). \qquad (3.5)$$

Under $H_1$ of (3.4) we have $(\mathbf{Y}|\beta,\delta,k,l) \sim \mathcal{N}(\mathbf{X}\beta + \delta\mathbf{S}_{kl,m}, \sigma^2\mathbf{I})$, where $\mathbf{S}_{kl,m} = \sum_{j=0}^{m} \mathbf{X}_{k,j}^{l}$. As above using the prior of $\beta$ we conclude that $(\mathbf{Y}|\delta,k,l) \sim \mathcal{N}(\delta S_{k,l}^{m}, \Sigma_\circ)$.

Let $c = (2\pi)^{-\frac{n}{2}} \mid \Sigma_\circ \mid^{-\frac{1}{2}}$, then under $H_1$ the Likelihood function (after integrating out $\delta$) is;

$$\begin{aligned}
& L_1(\mathbf{Y}|k,l) \\
= & \; c\int_{-\infty}^{\infty} \theta^{-1}(2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\mathbf{Y}-\delta\mathbf{S}_{kl,m})'\Sigma_\circ^{-1}(\mathbf{Y}-\delta\mathbf{S}_{kl,m}) - \frac{\delta^2}{2\theta^2}\}\mathrm{d}\delta \\
= & \; L_\circ(\mathbf{Y})\int_{-\infty}^{\infty} \theta^{-1}(2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2\theta^2}(\delta^2 + \delta^2\theta^2\mathbf{S}'_{kl,m}\Sigma_\circ^{-1}\mathbf{S}_{kl,m} - 2\delta\theta^2\mathbf{Y}^*)\}\mathrm{d}\delta \\
= & \; L_\circ(\mathbf{Y})\int_{-\infty}^{\infty} \theta^{-1}(2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2\theta^2}(\delta^2(1 + \theta^2\mathbf{S}'_{kl,m}\Sigma_\circ^{-1}\mathbf{S}_{kl,m}) - 2\delta\theta^2\mathbf{Y}^*)\}\mathrm{d}\delta \\
= & \; L_\circ(\mathbf{Y})\, d^{-1}\int_{-\infty}^{\infty} d\,\theta^{-1}(2\pi)^{-\frac{1}{2}} \exp\{-\frac{d^2}{2\theta^2}(\delta - \frac{\theta^2}{d^2}\mathbf{Y}^*)^2 + \frac{d^2}{2\theta^2}((\frac{\theta}{d})^2\mathbf{Y}^*)^2\}\mathrm{d}\delta \\
= & \; L_\circ(\mathbf{Y})\, d^{-1} \exp\{\frac{1}{2}(\frac{\theta}{d})^2(\mathbf{Y}^{*\prime}\mathbf{Y}^*)\}, \qquad (3.6)
\end{aligned}$$

where $d^2 = (1 + \theta^2\mathbf{S}'_{kl,m}\Sigma_\circ^{-1}\mathbf{S}_{kl,m})$ and $\mathbf{Y}^* = \mathbf{S}'_{kl,m}\Sigma_\circ^{-1}\mathbf{Y}$. By (3.6) the unconditional Likelihood function under $H_1$ is;

$$L_1(\mathbf{Y}) = L_\circ(\mathbf{Y}) \sum_{0\le k<l\le n-1} P(k,l)\, d^{-1} \exp\{\frac{1}{2}(\frac{\theta}{d})^2(\mathbf{Y}^{*\prime}\mathbf{Y}^*)\}. \qquad (3.7)$$

Hence, the Bayesian Likelihood ratio is given by:

$$\begin{aligned}
\frac{L_1(\mathbf{Y})}{L_\circ(\mathbf{Y})} &= \sum_{0\le k<l\le n-1} P(k,l)\, d^{-1} \exp\{\frac{1}{2}(\frac{\theta}{d})^2(\mathbf{Y}^{*\prime}\mathbf{Y}^*)\} \\
&= \sum_{0\le k<l\le n-1} P(k,l)\, d^{-1}\{1 + \frac{1}{2}(\frac{\theta}{d})^2(\mathbf{Y}^{*\prime}\mathbf{Y}^*) + o(\theta^3)\}. \qquad (3.8)
\end{aligned}$$

Note that, $d \to 1$ as $\theta \to 0$ and as in Theorem (2.2) of Jandhyala and MacNeill (1991), Woodbury's formula with $\tau \to \infty$ implies that $\Sigma_\circ^{-1} = \sigma^{-2}\mathbf{R}$. Ignoring terms of order $o(\theta^3)$ we have;

$$\frac{L_1(\mathbf{Y})}{L_\circ(\mathbf{Y})} \simeq 1 + \frac{1}{2}\theta^2 \sum_{0\le k<l\le n-1} P(k,l)(\mathbf{Y}^{*\prime}\mathbf{Y}^*). \qquad (3.9)$$

Therefore, a statistic to test for the hypotheses in (3.4) can be based on the second term of the R.H.S of (3.9).

## 3.4 Distribution theory

Jandhyala and MacNeill (1989) pointed out that the asymptotic theory of statistics like $T_n$ of Theorem (3.1) above, which are quadratic forms in the regression residuals, is complicated and only tractable to some extent. As in their work we will discuss the asymptotic distribution of $T_n$ when the regressor functions $f_i(.)$, $i = 0, 1, \ldots, m$, are defined on $[0, 1]$ and the observations are equi-spaced.

Consider the regression model;

$$\mathbf{Y} = \mathbf{P}\beta + \varepsilon, \tag{3.10}$$

where the design matrix $\mathbf{P}$ is given by:

$$\mathbf{P} = \begin{pmatrix} f_0(\frac{1}{n}) & f_1(\frac{1}{n}) & f_2(\frac{1}{n}) & \ldots & \ldots & .. & f_m(\frac{1}{n}) \\ f_0(\frac{2}{n}) & f_1(\frac{2}{n}) & f_2(\frac{2}{n}) & \ldots & \ldots & .. & f_m(\frac{2}{n}) \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ f_0(\frac{n}{n}) & f_1(\frac{n}{n}) & f_2(\frac{n}{n}) & \ldots & \ldots & .. & f_m(\frac{n}{n}) \end{pmatrix}$$

with regressor functions $f_j(.)$, $j = 0, 1, \ldots, m$, defined on $[0, 1]$. The above formulation of the model and (3.5) enables us to obtain

$$\mathbf{S}_{kl,m} = (0, \ldots, 0, r(\frac{k+1}{n}), r(\frac{k+2}{n}), \ldots, r(\frac{l}{n}), 0, \ldots, 0)', \tag{3.11}$$

where $r(.) = \sum_{j=0}^{m} f_j(.)$, which is also defined on $[0, 1]$. Hence the statistic $T_n$ of Theorem (3.1) can be written as :

$$\begin{aligned} T_n &= \sum_{0 \leq k < l \leq n-1} P(k, l)\mathbf{Y}'\mathbf{R}\, \mathbf{S}_{kl,m}\mathbf{S}'_{kl,m}\mathbf{R}\; \mathbf{Y} \\ &= \sum_{0 \leq k < l \leq n-1} P(k, l)\; \mathbf{e}'\; \mathbf{S}_{kl,m}\; \mathbf{S}'_{kl,m}\; \mathbf{e}, \end{aligned} \tag{3.12}$$

where $\mathbf{e}' = (e_1, \ldots, e_n) = (y_1 - \hat{y}_1, \ldots, y_n - \hat{y}_n)$ is the regression residual vector.

By (3.11) and (3.12) we can see that:

$$\begin{aligned}
\hat{T}_n &= T_n/(n\sigma^2) \\
&= \sum_{0 \leq k < l \leq n-1} P(k,l)\,\{\sum_{i=k+1}^{l} r(\frac{i}{n})(y_i - \hat{y}_i)\}^2/(n\sigma^2) \\
&= \sum_{0 \leq k < l \leq n-1} P(k,l)\,\{\sum_{i=1}^{l} r(\frac{i}{n})(y_i - \hat{y}_i) - \sum_{i=1}^{k} r(\frac{i}{n})(y_i - \hat{y}_i)\}^2/(n\sigma^2).
\end{aligned} \tag{3.13}$$

Define the sequence of stochastic processes $\{\gamma_n^{(r)}(t), t \in [0,1]\}$, $n = 1, 2, \ldots$, by:

$$\sigma\sqrt{n}\gamma_n^{(r)}(t) = \sum_{i=1}^{[nt]} r(\frac{i}{n})\; e_i + (nt - [nt])\; r(\frac{[nt]+1}{n})\; e_{[nt]+1}, \tag{3.14}$$

where the $\epsilon_i$'s are as in (3.12) and $r(.)$ is as in (3.11). Observe that; $\sum_{i=1}^{l} r(\frac{i}{n})\; e_i = \sigma\sqrt{n}\gamma_n^{(r)}(\frac{l}{n})$. Let $\{W(t), t \in [0,1]\}$ be a standard Wiener process . Let

$$\mathbf{f}(t) = (f_o(t), f_1(t), \ldots, f_m(t))'. \tag{3.15}$$

Define $\mathbf{F}$ and $g_m(s,t)$ by :

$$\mathbf{F} = \lim_{n \to \infty} (\mathbf{P}'\mathbf{P})/n = \begin{pmatrix} \int_0^1 f_o^2(t)\mathrm{d}t & \cdots & \int_0^1 f_o(t) f_m(t)\mathrm{d}t \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ \int_0^1 f_m(t) f_o(t)\mathrm{d}t & \cdots & \int_0^1 f_m^2(t)\mathrm{d}t \end{pmatrix}$$

and, for $0 \leq s, t \leq 1$,

$$g_m(s,t) = \mathbf{f}'(s)\mathbf{F}^{-1}\mathbf{f}(t). \tag{3.16}$$

In our results we need the following lemma of Jandhyala and MacNeill (1989).

**Lemma** (3.A):

Let the regressor functions $f_i(t)$, $t \in [0,1]$, $i = 0, 1, \ldots, m$ and $r(.)$ be continuously differentiable on $[0,1]$. Then the sequence of stochastic processes $\{\gamma_n^{(r)}(t), t \in [0,1]\}$ $n = 1, 2, \ldots,$ converges weakly to the Gaussian process $\{W_m^{(r)}(t), t \in [0,1]\}$ defined by

$$W_m^{(r)}(t) = \int_0^t r(x)\mathrm{d}W(x) - \int_0^t r(x)\{\int_0^1 g_m(x,y)\mathrm{d}W(y)\}\mathrm{d}x, \qquad (3.17)$$

with mean zero and covariance function given by

$$\begin{aligned} K_m^{(r)}(s,t) &= Cov(W_m^{(r)}(s), W_m^{(r)}(t)) \\ &= \int_0^{(s\wedge t)} r(x)^2 \mathrm{d}x - \int_0^s \int_0^t r(x)r(y)g_m(x,y)\mathrm{d}x\mathrm{d}y. \qquad (3.18) \end{aligned}$$

Let $\psi(.,.)$ be a non-negative continuous weight function, such that:

$$P(k,l) = \int_{\frac{l}{n}}^{\frac{l+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(x,y)\mathrm{d}x\mathrm{d}y, \qquad (3.19)$$

and

$$\sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \int_{\frac{l}{n}}^{\frac{l+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(x,y)\mathrm{d}x\mathrm{d}y < \infty. \qquad (3.20)$$

**Lemma** (3.1)

Let $\hat{D}_n$ and $D_n$ be defined by;

$$\hat{D}_n = \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \int_{\frac{l}{n}}^{\frac{l+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(x,y)\{\gamma_n^{(r)}(\frac{l}{n}) - \gamma_n^{(r)}(\frac{k}{n})\}^2 \mathrm{d}x\mathrm{d}y.$$

and;

$$D_n = \int_0^1 \int_0^y \psi(x,y)\{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y.$$

Then,

$$| \hat{D}_n - D_n | \xrightarrow{P} 0,$$

where $\gamma_n^{(r)}(.)$ is defined by (3.14) and $\psi(.,.)$ satisfies (3.19) and (3.20).

**Proof**

By Markov inequality we obtain for any $\eta > 0$;

$$P(| \hat{D}_n - D_n | > \eta) \leq E | \hat{D}_n - D_n | / \eta. \tag{3.21}$$

Let

$$\hat{d}^2_{k,l}(n) = \{\gamma_n^{(r)}(\frac{l}{n}) - \gamma_n^{(r)}(\frac{k}{n})\}^2, \qquad 0 \leq k < l \leq n-1, \tag{3.22}$$

$$d^2_{x,y}(n) = \{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2, \qquad 0 \leq x < y \leq 1, \tag{3.23}$$

and; for $0 \leq k < l \leq n-1$ and $0 \leq x < y \leq 1$,

$$\theta(n) = | \hat{d}^2_{k,l}(n) - d^2_{x,y}(n) | . \tag{3.24}$$

Using (3.22)-(3.24) we have,

$$\begin{aligned} E | \hat{D}_n - D_n | \leq \quad & E \sum_{l=1}^{n-1} \sum_{k=0}^{l-1} \int_{\frac{l}{n}}^{\frac{l+1}{n}} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(x,y)\theta(n) \mathrm{d}x\mathrm{d}y \\ & + E \int_0^{\frac{1}{n}} \int_0^y \psi(x,y)\{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y \\ & + E \sum_{l=1}^{n-1} \int_{\frac{l}{n}}^{\frac{l+1}{n}} \int_{\frac{l}{n}}^{y} \psi(x,y)\{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.25}$$

Consider the first term of (3.25). For $\frac{k}{n} \leq x < \frac{k+1}{n}$ and $\frac{l}{n} \leq y < \frac{l+1}{n}$, we obtain

$$\begin{aligned} E\theta(n) & = E | (\hat{d}_{k,l}(n) - d_{x,y}(n))(\hat{d}_{k,l}(n) + d_{x,y}(n)) | \\ & \leq \{E(\Delta_1(n) - \Delta_2(n))^2 E(\Delta_3(n) - \Delta_4(n))^2\}^{\frac{1}{2}}, \end{aligned} \tag{3.26}$$

where, $\Delta_1(n) = \gamma_n^{(r)}(\frac{l}{n}) - \gamma_n^{(r)}(y)$, $\Delta_2(n) = \gamma_n^{(r)}(\frac{k}{n}) - \gamma_n^{(r)}(x)$, $\Delta_3(n) = \gamma_n^{(r)}(\frac{l}{n}) + \gamma_n^{(r)}(y)$, and $\Delta_4(n) = \gamma_n^{(r)}(\frac{k}{n}) + \gamma_n^{(r)}(x)$. By the definition of the process $\gamma_n^{(r)}(.)$ in (3.14) and when $\frac{l}{n} \leq y < \frac{l+1}{n}$, i.e. $l \leq ny < (l+1)$ we have,

$$\Delta_1(n) = \frac{1}{\sigma\sqrt{n}}(l - ny)r(\frac{l+1}{n})e_{l+1},$$

and hence,

$$E\Delta_1^2(n) = \frac{1}{\sigma^2 n}(l - ny)^2 r^2(\frac{l+1}{n})Ee_{l+1}^2. \tag{3.27}$$

Now for the residual vector $\mathbf{e} = (e_1, \ldots, e_n)'$ we have:

$$Var(\mathbf{e}) = \mathbf{R}\ Var(\mathbf{Y})\ \mathbf{R}' = \sigma^2\ \mathbf{R}, \tag{3.28}$$

where; $\mathbf{R} = \mathbf{I} - \mathbf{P}(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}'$ and $\mathbf{P}$ is the design matrix of the model (3.10). Note that by the definition of the matrix $\mathbf{P}$ and the matrix $\mathbf{F}$ in (3.16), the components of the matrix $\mathbf{P}(\mathbf{P}'\mathbf{P})^{-1}\mathbf{P}' = (\ a_{ij}\ )_{1\leq i,j\leq n}$ are such that:

$$|\ a_{ij}\ |= O(\frac{1}{n}), \qquad \forall i, j. \tag{3.29}$$

Thus by (3.28) and (3.29) we have for any $1 \leq i, j \leq n$,

$$|\ E(e_i e_j)\ |= \begin{cases} \sigma^2\ O(\frac{1}{n}) & \text{for } i \neq j \\ \sigma^2\ (1 - O(\frac{1}{n})) & \text{for } i = j \end{cases}, \tag{3.30}$$

The result of (3.30) above requires the regression residuals to behave like the regression errors (at least approximately) as $n$ increase. Using (3.27) and (3.30) we obtain by the boundedness of $r(.)$ and as $n \to \infty$,

$$\begin{aligned} E(\Delta_1^2(n)) &\leq \frac{1}{\sigma^2 n} \max_{1\leq i\leq n} r^2(\frac{i}{n}) \max_{1\leq i\leq n} E(e_i^2) \\ &= O(\frac{1}{n}). \end{aligned} \tag{3.31}$$

Similarly, for $\Delta_2(n)$ of (3.26), when $\frac{k}{n} \leq y < \frac{k+1}{n}$, i.e. $k \leq nx < (k+1)$,we have as $n \to \infty$,

$$\begin{aligned} E(\Delta_2^2(n)) &\leq \frac{1}{\sigma^2 n} \max_{1 \leq i \leq n} r^2(\frac{i}{n}) \max_{1 \leq i \leq n} E(e_i^2) \\ &= O(\frac{1}{n}). \end{aligned} \tag{3.32}$$

For the cross-product term $\Delta_1(n)\Delta_2(n)$ in (3.26), when $\frac{l}{n} \leq y < \frac{l+1}{n}$ and $\frac{k}{n} \leq y < \frac{k+1}{n}$, we have by (3.30) and as $n \to \infty$,

$$\begin{aligned} E(\Delta_1(n)\Delta_2(n)) &= E\{\frac{1}{\sigma\sqrt{n}}(l - ny)\, r(\frac{l+1}{n})e_{l+1}\}\{\frac{1}{\sigma\sqrt{n}}(k - nx)\, r(\frac{k+1}{n})e_{k+1}\} \\ &\leq \frac{1}{\sigma^2 n} \mid r(\frac{l+1}{n})r(\frac{k+1}{n}) \mid \{\mid E(e_{l+1}e_{k+1}) \mid\} \\ &\leq \frac{1}{\sigma^2 n}\{\max_{1 \leq i \leq n} \mid r(\frac{i}{n}) \mid\}^2 \{E \mid e_{l+1} \mid\mid e_{k+1} \mid\} \\ &\leq \frac{1}{\sigma^2 n}\{\max_{1 \leq i \leq n} \mid r(\frac{i}{n}) \mid\}^2 \{Ee_{l+1}^2 Ee_{k+1}^2\}^{\frac{1}{2}} \\ &\leq \frac{1}{\sigma^2 n}\{\max_{1 \leq i \leq n} \mid r(\frac{i}{n}) \mid\}^2 \max_{1 \leq i \leq n} Ee_i^2 \\ &= O(\frac{1}{n}). \end{aligned} \tag{3.33}$$

We note, by the boundedness of $r(.)$ and (3.30), that as $n \to \infty$,

$$\begin{aligned} E_l^2 &= E(\sum_{i=1}^{l} r(\frac{i}{n})e_i)^2 \\ &= \sum_{i=1}^{l}\sum_{j=1}^{l} r(\frac{i}{n})\, r(\frac{j}{n})\, E(e_i e_j) \\ &\leq \sum_{i=1}^{l}\sum_{j=1}^{l} \mid r(\frac{i}{n})\, r(\frac{j}{n}) \mid\mid E(e_i e_j) \mid \\ &\leq \{\max_{1 \leq i \leq n} \mid r(\frac{i}{n}) \mid\}^2 \sum_{i=1}^{l}\sum_{j=1}^{l} \mid E(e_i e_j) \mid \\ &\leq \{\max_{1 \leq i \leq n} \mid r(\frac{i}{n}) \mid\}^2 \{\sum_{i=1}^{n} E(e_i^2) + \sum_{i \neq j}^{n} \mid E(e_i e_j) \mid\} \end{aligned}$$

$$\begin{aligned} &\leq \{\max_{1\leq i\leq n} \mid r(\frac{i}{n}) \mid\}^2\{n \max_{1\leq i\leq n} E(e_i^2) + n(n-1) \max_{1\leq i,j\leq n} \mid E(e_ie_j) \mid\} \\ &\leq \sigma^2 O(n). \qquad (3.34) \end{aligned}$$

For the square term $\Delta_3^2(n)$, (or $\Delta_4^2(n)$) in (3.26), we have by (3.30) and (3.34) , when $\frac{l}{n} \leq y < \frac{l+1}{n}$, (or $\frac{k}{n} \leq y < \frac{k+1}{n}$) and as $n \to \infty$,

$$\begin{aligned} E\Delta_3^2(n) &= \frac{1}{\sigma^2 n} E\{2\sum_{i=1}^{l} r(\frac{i}{n})\, e_i + (l-ny)\, r(\frac{l+1}{n})\, e_{l+1}\}^2 \\ &= \frac{1}{\sigma^2 n}\, \{4\, E_l^2 + (l-ny)^2\, r^2(\frac{l+1}{n})\, Ee_{l+1}^2 \\ &+ 4\,(l-ny)\, r(\frac{l+1}{n}) \sum_{i=1}^{l} r(\frac{i}{n})\, E(e_ie_{l+1})\} \\ &\leq \frac{1}{\sigma^2 n}\, \{4\, E_l^2 + \max_{1\leq i\leq n} r^2(\frac{i}{n}) \max_{1\leq i\leq n} E(e_i^2) \\ &+ 4\,\{\max_{1\leq i\leq n} \mid r(\frac{i}{n}) \mid\}^2\, n \max_{1\leq i\leq n} \mid E(e_ie_j) \mid\} \\ &\leq O(1). \qquad (3.35) \end{aligned}$$

As in (3.34) we can easily see that when $n \to \infty$,

$$E_{kl} = E(\sum_{i=1}^{l}\sum_{j=1}^{k} r(\frac{i}{n})\, r(\frac{j}{n})\, e_ie_j) = O(n). \qquad (3.36)$$

Hence for the cross-product term $\Delta_3(n)\Delta_4(n)$ of (3.26), when $\frac{l}{n} \leq y < \frac{l+1}{n}$, and $\frac{k}{n} \leq y < \frac{k+1}{n}$ we have as $n \to \infty$,

$$\begin{aligned} E(\Delta_3(n)\Delta_4(n)) &= \frac{1}{\sigma^2 n} E\{2\sum_{i=1}^{l} r(\frac{i}{n})\, e_i + (l-ny)\, r(\frac{l+1}{n})\, e_{l+1}\} \\ &\quad \{2\sum_{j=1}^{k} r(\frac{j}{n})\, e_j + (l-ny)\, r(\frac{l+1}{n})\, e_{l+1}\}, \\ &\leq O(1). \qquad (3.37) \end{aligned}$$

Now, for the second term of (3.25) we get by the definition of $\psi(.,.)$ and as $n \to \infty$,

$$\begin{aligned} t_1 &= E\int_0^{\frac{1}{n}}\int_0^y \psi(x,y)\{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y \\ &\leq \sup_{0 \leq x < y < 1} \psi(x,y) E\int_0^{\frac{1}{n}}\int_0^y \{\frac{ny}{\sigma\sqrt{n}} r(\frac{1}{n})e_1 - \frac{nx}{\sigma\sqrt{n}} r(\frac{1}{n})e_1\}^2 \mathrm{d}x\mathrm{d}y \\ &\leq \sup_{0 \leq x < y < 1} \psi(x,y) \frac{n}{\sigma^2} r^2(\frac{1}{n}) \int_0^{\frac{1}{n}}\int_0^y (y-x)^2 \mathrm{d}x\mathrm{d}y Ee_1^2 \\ &\leq \sup_{0 \leq x < y < 1} \psi(x,y) \frac{n}{\sigma^2} r^2(\frac{1}{n})(\frac{1}{12n^4})\ Ee_1^2 \\ &\leq O(\frac{1}{n^3}). \end{aligned} \tag{3.38}$$

Also, for the third term of (3.25) we have as $n \to \infty$,

$$\begin{aligned} t_2 &= E\sum_{l=1}^{n-1}\int_{\frac{l}{n}}^{\frac{l+1}{n}}\int_{\frac{l}{n}}^y \psi(x,y)\{\gamma_n^{(r)}(y) - \gamma_n^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y \\ &\leq \sup_{0 \leq x < y < 1} \psi(x,y) E\sum_{l=1}^{n-1}\int_{\frac{l}{n}}^{\frac{l+1}{n}}\int_{\frac{l}{n}}^y \{\frac{ny}{\sigma\sqrt{n}} r(\frac{l+1}{n})e_{l+1} - \frac{nx}{\sigma\sqrt{n}} r(\frac{l+1}{n})e_{l+1}\}^2 \mathrm{d}x\mathrm{d}y \\ &\leq \frac{n}{\sigma^2} \sup_{0 \leq x < y < 1} \psi(x,y)\{\max_{1 \leq i \leq n} r^2(\frac{i}{n})\}\{\max_{1 \leq i \leq n} Ee_i^2\} \sum_{l=1}^{n-1}\int_{\frac{l}{n}}^{\frac{l+1}{n}}\int_{\frac{l}{n}}^y (y-x)^2 \mathrm{d}x\mathrm{d}y \\ &\leq O(\frac{1}{n^2}). \end{aligned} \tag{3.39}$$

Using (3.31)-(3.33), (3.35) and (3.37)-(3.39) in (3.25) we get for any $\eta > 0$ , as $n \to \infty$,

$$\begin{aligned} P & \quad (\mid \hat{D}_n - D_n \mid > \eta) \\ &\leq E \mid \hat{D}_n - D_n \mid / \eta, \\ &\leq (\frac{1}{\eta}) O(\frac{1}{\sqrt{n}}) \sum_{l=1}^{n-1}\sum_{k=0}^{l-1} \int_{\frac{l}{n}}^{\frac{l+1}{n}}\int_{\frac{k}{n}}^{\frac{k+1}{n}} \psi(x,y)\mathrm{d}x\mathrm{d}y + O(\frac{1}{n^3}) + O(\frac{1}{n^2}), \\ &\leq (\frac{1}{\eta}) O(\frac{1}{\sqrt{n}}) \to 0, \end{aligned} \tag{3.40}$$

and hence, $\mid \hat{D}_n - D_n \mid \xrightarrow{p} 0$ as $n \to \infty$.

**Lemma** (3.2)

Let $\psi(x,y)$ be continuous on $0 \leq x < y \leq 1$ and $r(t)$ be a continuously differentiable function on $0 \leq t \leq 1$. Then,

$$\lim_{n\to\infty} P\{Z_\psi(\gamma_n^{(r)}) \leq \alpha\} = P\{Z_\psi(W_m^{(r)}) \leq \alpha\},$$

uniformly in $\alpha$, where $\gamma_n^{(r)}(.)$ and $W_m^{(r)}(.)$ are defined by (3.14) and (3.17) respectively and;

$$Z_\psi(f) = \int_0^1 \int_0^y \psi(x,y)\{f(y) - f(x)\}^2 \mathrm{d}x\mathrm{d}y.$$

The proof of this Lemma is an immediate result of Theorem (5.2) of Billingsley (1968) and Lemma (2.1) of Jandhyala and MacNeill (1989).

### 3.5 Applications

In this section we will study the asymptotic distribution of the test $\hat{T}_n$ of (3.13) assuming that the weight function of (3.20); $\psi(x,y) = C > 0$ (a constant), $0 \leq x < y \leq 1$. This assumption will simplify the asymptotic distribution of $\hat{T}_n$, since in this case $\hat{T}_n \xrightarrow{d} T$, where

$$\begin{aligned} T = C & \quad \int_0^1 \int_0^y \{W_m^{(r)}(y) - W_m^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y, \\ = C & \quad \{ \int_0^1 \int_0^y \{W_m^{(r)}(y)\}^2 \mathrm{d}x\mathrm{d}y + \int_0^1 \int_0^y \{W_m^{(r)}(x)\}^2 \mathrm{d}x\mathrm{d}y \\ & \quad -2\int_0^1 \int_0^y W_m^{(r)}(y) W_m^{(r)}(x) \mathrm{d}x\mathrm{d}y\}, \end{aligned}$$

$$\begin{aligned}
&= C \quad \{ \quad \int_0^1 y\{W_m^{(r)}(y)\}^2 \mathrm{d}y + \int_0^1 (1-x)\{W_m^{(r)}(x)\}^2 \mathrm{d}x - (\int_0^1 \{W_m^{(r)}(x)\mathrm{d}x)^2\}, \\
&= C \quad \{ \quad \int_0^1 \{W_m^{(r)}(y)\}^2 \mathrm{d}y - (\int_0^1 \{W_m^{(r)}(x)\}\mathrm{d}x)^2\}, \\
&= C \quad \{ \quad \int_0^1 (G_m^{(r)}(x))^2 \mathrm{d}x\}, \qquad (3.41) \\
G_m^{(r)}(x) \quad &= \quad W_m^{(r)}(x) - \int_0^1 W_m^{(r)}(x)\mathrm{d}x
\end{aligned}$$

and $W_m^{(r)}(.)$ is as in (3.17).

Next we give some examples.

**Example 1:**

Consider the case of fitting a mean to a set of data, i.e.

$$\mathbf{Y} = \beta_\circ \mathbf{1} + \epsilon, \qquad (3.42)$$

where $\mathbf{1} = (1, \ldots, 1)'$. Under this model, we can test for an epidemic-type change in the intercept parameter $\beta_\circ$. In this case $T$ of (3.41) becomes;

$$\begin{aligned}
T^\circ &= C \int_0^1 (G_\circ^{(1)}(x))^2 \mathrm{d}x, \\
&= C \int_0^1 \{W_\circ^{(1)}(x) - \int_0^1 W_\circ^{(1)}(y)\mathrm{d}y\}^2 \mathrm{d}x,
\end{aligned}$$

where, by (3.17), $W_\circ^{(1)}(t) = W(t) - tW(1)$, $t \in [0,1]$, is a standard Brownian bridge. Watson (1961) showed that;

$$P(T^\circ \leq C\nu) = 1 - 2\sum_{j=1}^{\infty}(-1)^{j-1}e^{-2j^2\pi^2\nu}.$$

**Example 2:**

Suppose we are still investigating a possible epidemic-type change in the intercept parameter $\beta_o$, but this time considering a simple linear regression model given by;

$$\mathbf{Y} = \mathbf{P}\beta + \epsilon, \tag{3.43}$$

where $\beta' = (\beta_o \;\; \beta_1)$, the $i^{th}$ row of the design matrix $\mathbf{P}$ is $(1 \;\; f(\frac{i}{n}))$, $i = 1, \ldots, n$ and $f(x) = x$. Testing for an epidemic-type change in $\beta_o$ of the model in (3.43), we note that the limiting distribution in (3.41), reduces to;

$$\begin{aligned} T^1 &= C\int_0^1 (G_1^{(1)}(x))^2 \mathrm{d}x, \\ &= C\int_0^1 \{W_1^{(1)}(x) - \int_0^1 W_1^{(1)}(u)\mathrm{d}u\}^2 \mathrm{d}x, \end{aligned} \tag{3.44}$$

where $W_1^{(1)}$ is defined by (3.17) with $r(x) = 1$,

$$\begin{aligned} g_1(x,y) &= \mathbf{f}'(x)\,\mathbf{F}^{-1}\mathbf{f}(y), \\ &= (1 \;\; x)\mathbf{F}^{-1}(1 \;\; y)', \\ &= 12(\frac{1}{3} - \frac{x}{2} - \frac{y}{2} + xy), \qquad 0 < x, y < 1, \end{aligned} \tag{3.45}$$

and the matrix $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} 1 & \int_0^1 t\mathrm{d}t \\ \int_0^1 t\mathrm{d}t & \int_0^1 t^2\mathrm{d}t \end{pmatrix}.$$

Substituting (3.45) in (3.18) we find that the covariance function of $W_1^{(1)}(.)$ is given by

$$\begin{aligned} K_1^{(1)}(s,t) &= \int_0^{s\wedge t} r^2(x)\mathrm{d}x - \int_0^s \int_0^t r(x)r(y)g_1(x,y)\mathrm{d}x\mathrm{d}y \\ &= (s\wedge t) - ts(4 - 3t - 3s + 3st), \qquad 0 < s,t < 1. \end{aligned} \tag{3.46}$$

Let $W_m^{(r)}(x)$, $0 < x < 1$ be defined by (3.17) with covariance function $K_m^{(r)}(x,y)$, $0 < x, y < 1$ as in (3.18), then the covariance function of $G_m^{(r)}(x)$, $0 < x < 1$ of (3.41), becomes

$$\begin{aligned} Q_m^{(r)}(s,t) &= K_m^{(r)}(s,t) - \int_0^1 K_m^{(r)}(y,t)\mathrm{d}y - \int_0^1 K_m^{(r)}(x,s)\mathrm{d}x \\ &\quad + \int_0^1 \int_0^1 K_m^{(r)}(x,y)\mathrm{d}x\mathrm{d}y, \qquad (3.47) \end{aligned}$$

where $0 < s, t < 1$. From (3.46) and (3.47), the covariance function of $G_1^{(1)}(.)$ of (3.44) is;

$$\begin{aligned} Q_1^{(1)}(s,t) &= K_1^{(1)}(s,t) \\ &= (s \wedge t) - ts(4 - 3t - 3s + 3st), \qquad 0 < s, t < 1. \qquad (3.48) \end{aligned}$$

To specify the distribution of the stochastic integral involved in $T^1$ of (3.44), we will use Anderson and Darling (1952) method. They applied this method to obtain the distribution of a functional of a Brownian bridge. MacNeill(1974 and 1978) adopted it to find the distributions of functionals of a Brownian motion and a Brownian bridge (for more details we refer to Shorack and Wellner (1986)). This technique requires the expansion of the process $G_1^{(1)}(.)$ as a weighted sum of uncorrelated, zero mean Normal random variables $\{Z_n\}_{n=1}^{\infty}$ satisfying,

$$E\{G_1^{(1)}(x) - \sum_{i=1}^{\infty} Z_i h_i(x)\}^2 = 0, \qquad 0 < x < 1.$$

where, $Var(Z_i) = \lambda_i$, $i = 1, \dots$ and $\{h_i(.)\}_{i=1}^{\infty}$ is a set of functions satisfying

$$\int_0^1 Q_1^{(1)}(s,t)h_i(s)\mathrm{d}s = \lambda_i h_i(t), \qquad i = 1, 2, \dots \qquad (3.49)$$

and,

$$\int_0^1 h_i(x)h_j(x)\mathrm{d}x = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j \end{cases} . \tag{3.50}$$

Hence, by (3.48) and (3.49) we have for $n = 1, 2, \ldots$

$$\begin{aligned} \lambda_n h_n(t) &= \int_0^1 \{(s \wedge t) - ts(4 - 3t - 3s + 3st)\} h_n(s)\mathrm{d}s, \\ &= \int_0^t s\, h_n(s)\mathrm{d}s + \int_t^1 t\, h_n(s)\mathrm{d}s - 4t \int_0^1 s\, h_n(s)\mathrm{d}s \\ &\quad +3t^2 \int_0^1 s\, h_n(s)\mathrm{d}s + 3t \int_0^1 s^2 h_n(s)\mathrm{d}s - 3t^2 \int_0^1 s^2 h_n(s)\mathrm{d}s. \end{aligned} \tag{3.51}$$

Differentiating (3.51) three times w.r.t., t, we get;

$$\lambda_n h_n^{'}(t) = \int_t^1 h_n(s)\mathrm{d}s + (6t - 4)\int_0^1 s\, h_n(s)\mathrm{d}s + (3 - 6t)\int_0^1 s^2 h_n(s)\mathrm{d}s, \tag{3.52}$$

$$\lambda_n h_n^{''}(t) = -h_n(t) + 6\int_0^1 s\, h_n(s)\mathrm{d}s - 6\int_0^1 s^2 h_n(s)\mathrm{d}s, \tag{3.53}$$

$$\lambda_n h_n^{'''}(t) = -h_n^{'}(t). \tag{3.54}$$

MacNeill (1978), solved the differential equation in (3.54) under the boundary conditions; $h_n(0) = h_n(1) = 0$ and $\lambda_n h_n^{''}(1) = 6\int_0^1 s(1-s)\, h_n(s)\mathrm{d}s$, $n = 1, 2, \ldots$. He also gave a table of selected quantiles for the corresponding distribution when $C = 1$.

**Example 3:**

Consider the problem of testing for a possible epidemic-type change in the slope parameter of the model described in Example 2. For simplicity,we assume here that there is no intercept term in the model, i.e. $\beta_o = 0$. With the above formulation in mind, and by (3.11), (3.16), (3.18) and (3.47) it is easy to see that;

$$r(x) = x,$$

$$g_1(u,v) = u(\int_0^1 t^2 \mathrm{d}t)^{-1} v = 3uv,$$

$$K_1^{(x)}(s,t) = \int_0^{s\wedge t} u^2 \mathrm{d}u - \int_0^s \int_0^t uv(3uv)\mathrm{d}u\mathrm{d}v = \frac{1}{3}\{(s\wedge t)^3 - (ts)^3\},$$

and,

$$Q_1^{(x)}(s,t) = \frac{1}{3}\{(s\wedge t)^3 - (ts)^3\} - \frac{t^3}{4}(1-t) - \frac{s^3}{4}(1-s) + \frac{1}{80}. \qquad (3.55)$$

Using Anderson-Darling technique as in Example 2 to determine the distribution corresponding to (3.55) we need to solve the following equation;

$$\begin{aligned}\lambda_n h_n(t) &= \int_0^1 Q_1^{(x)}(s,t) h_n(s)\mathrm{d}s \\ &= \int_0^1 \{\frac{1}{3}(s\wedge t)^3 - \frac{1}{3}(st)^3 - \frac{t^3}{4}(1-t) - \frac{s^3}{4}(1-s) + \frac{1}{80}\} h_n(s)\mathrm{d}s.\end{aligned} \qquad (3.56)$$

Differentiating (3.56) four times w.r.t., t, we get;

$$\lambda_n h_n^{(4)}(t) + t^2 h_n^{(2)}(t) + 6t h_n^{(1)}(t) + 6h_n(t) = 0. \qquad (3.57)$$

The above differential equation is very hard to solve even when we try to assume an infinite series as a solution for $h_n(.)$. Alternatively, we calculate next empirical and approximate critical values for the test in this case.

Choosing $\psi(x,y)=2$, in (3.13), the BLR test for the model here reduces to;

$$\hat{T}_n = \frac{2}{\sigma^2 n^5}\sum_{k<l}\{\sum_{m=k+1}^{l} m\,\epsilon_m - \frac{l(l+1)(2l+1)-k(k+1)(2k+1)}{n(n+1)(2n+1)}\sum_{m=1}^{n} m\,\epsilon_m\}^2,$$

where $k$ and $l$ are the change positions and $\epsilon_i$, $i=1,\ldots,n$ are iid Normal with mean zero and variance $\sigma^2$,(taken to be =1). Based on Monte Carlo simulations we estimated the critical values of the test $\hat{T}_n$ for sample sizes $n=10,20,\ldots,100$. For each sample size $n$ we calculated the above test $5,000$ times. Then we ordered the $5,000$ values and obtained the $(1-\alpha)^{th}$ percentiles for $\alpha=0.1,0.05,0.01$. Table 1 gives the estimated critical values of $\hat{T}_n$. We can see from the values of Table 1, that the test $\hat{T}_n$ converges to a limit. To study the applicability of the estimated critical values in finite samples we conducted a Monte Carlo simulation to estimate the critical values of the distribution of (3.41) of Example 3. We generated a vector $Z=(z_1,\ldots,z_M)$ of multivariate Normal variates, with mean zero and covariance function $Q_1^{(x)}(.,.)$ of (3.55). This vector is a discrete trajectory "in distribution" version of the process $G_1^{(x)}(t)$, $0\le t\le 1$ of (3.41) at $t_i=\frac{i}{M+1}$, $i=1,\ldots,M$. We took $M=800$ and used $1,000$ realizations. For each of these realizations we calculated the quantity $C*\sum_{i=1}^{M}(z_i)^2/M$, as an estimate of the required integral. We then ordered the resulting $1,000$ values and obtained their $(1-\alpha)^{th}$ percentiles for $\alpha=0.1,0.05$ and $0.01$. These approximated (App.) quantiles are given in the last row of Table 1. The results of this Table show the closeness of the estimated critical values of the test statistic in small samples and the approximations of the limiting quantiles.

To study the power of the test under the alternative hypothesis of possible epidemic-type change, a Monte Carlo power study was performed. In this study, we set the sample size $n$ to 100. Several combinations of $k$ and $l$ were considered with two different change sizes $\delta = 0.5$ and $\delta = 1.0$. For each $(k, l)$ and $\delta$, we used $3,000$ replications. In each of these replications we calculated the test under the alternative hypothesis, i.e., under the changed model. Then we obtained the percentages of times that the test value exceeded the estimated critical value. The resulting power estimates are shown in Tables 2 and 3. We can see from the power Tables that the larger the change the larger the powers get. Also, the powers increase as the change positions move away from each other, but again decrease if either or both change positions are close to the end of the data set. The later decrease in power is not surprising since when either or both change positions are close to the end of the data set the test behaves as if there is only one or no change, respectively.

Table 1

Estimated critical values of the Bayesian test for an epidemic change in the slope of a simple regression model

| $n$ | $\alpha = 0.1$ | $\alpha = 0.05$ | $\alpha = 0.01$ |
|---|---|---|---|
| 10 | 0.0959 | 0.1294 | 0.2137 |
| 20 | 0.0913 | 0.1186 | 0.1767 |
| 30 | 0.0896 | 0.1164 | 0.1854 |
| 40 | 0.0897 | 0.1149 | 0.1754 |
| 50 | 0.0864 | 0.1098 | 0.1668 |
| 60 | 0.0862 | 0.1086 | 0.1691 |
| 70 | 0.0895 | 0.1138 | 0.1671 |
| 80 | 0.0899 | 0.1139 | 0.1695 |
| 90 | 0.0866 | 0.1106 | 0.1662 |
| 100 | 0.0884 | 0.1107 | 0.1667 |
| *App.* | 0.0831 | 0.1056 | 0.1523 |

Table 2

Estimated power percentages for the Bayesian-type test for an epidemic change in the slope of a simple linear regression model

at $\alpha = 0.05, n = 100, \delta = 0.5$

| $k$ | $l$ | *Est. power* | $k$ | $l$ | *Est. power* |
|---|---|---|---|---|---|
| 5 | 15 | 7.1 | 35 | 45 | 17.4 |
|  | 25 | 9.5 |  | 55 | 25.4 |
|  | 35 | 12.7 |  | 65 | 33.6 |
|  | 45 | 19.0 |  | 75 | 41.4 |
|  | 55 | 27.2 |  | 85 | 42.4 |
|  | 65 | 33.6 |  | 95 | 31.5 |
|  | 75 | 43.6 |  |  |  |
|  | 85 | 49.2 | 45 | 55 | 25.9 |
|  | 95 | 48.1 |  | 65 | 35.6 |
|  |  |  |  | 75 | 39.2 |
| 15 | 25 | 8.8 |  | 85 | 38.9 |
|  | 35 | 11.9 |  | 95 | 26.8 |
|  | 45 | 17.8 |  |  |  |
|  | 55 | 25.6 | 55 | 65 | 35.5 |
|  | 65 | 33.1 |  | 75 | 39.9 |
|  | 75 | 42.9 |  | 85 | 36.0 |
|  | 85 | 44.6 |  | 95 | 23.7 |
|  | 95 | 42.6 |  |  |  |
|  |  |  | 65 | 75 | 41.0 |
| 25 | 35 | 12.1 |  | 85 | 35.5 |
|  | 45 | 18.5 |  | 95 | 16.1 |
|  | 55 | 24.6 |  |  |  |
|  | 65 | 34.8 | 75 | 85 | 32.3 |
|  | 75 | 42.3 |  | 95 | 13.9 |
|  | 85 | 43.3 |  |  |  |
|  | 95 | 36.5 | 85 | 95 | 11.8 |

Table 3

Estimated power percentages for the Bayesian-type test for an epidemic change in the slope of a simple linear regression model at $\alpha = 0.05, n = 100, \delta = 1.0$

| $k$ | $l$ | *Est. power* | $k$ | $l$ | *Est. power* |
|---|---|---|---|---|---|
| 5 | 15 | 12.7 | 35 | 45 | 58.9 |
|  | 25 | 24.2 |  | 55 | 79.1 |
|  | 35 | 41.1 |  | 65 | 90.0 |
|  | 45 | 59.2 |  | 75 | 93.6 |
|  | 55 | 77.1 |  | 85 | 94.9 |
|  | 65 | 89.1 |  | 95 | 84.6 |
|  | 75 | 95.1 |  |  |  |
|  | 85 | 97.9 | 45 | 55 | 78.5 |
|  | 95 | 97.2 |  | 65 | 91.1 |
|  |  |  |  | 75 | 93.9 |
| 15 | 25 | 22.9 |  | 85 | 93.1 |
|  | 35 | 38.6 |  | 95 | 77.7 |
|  | 45 | 59.8 |  |  |  |
|  | 55 | 76.8 | 55 | 65 | 91.1 |
|  | 65 | 87.4 |  | 75 | 93.4 |
|  | 75 | 94.9 |  | 85 | 90.6 |
|  | 85 | 96.6 |  | 95 | 65.7 |
|  | 95 | 96.0 |  |  |  |
|  |  |  | 65 | 75 | 93.9 |
| 25 | 35 | 37.5 |  | 85 | 89.4 |
|  | 45 | 58.3 |  | 95 | 54.2 |
|  | 55 | 74.6 |  |  |  |
|  | 65 | 88.9 | 75 | 85 | 86.5 |
|  | 75 | 94.5 |  | 95 | 40.8 |
|  | 85 | 94.8 |  |  |  |
|  | 95 | 91.0 | 85 | 95 | 31.4 |

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# Chapter 4

# Rank tests in samples with random size

## 4.1 Introduction

Inferences drawn from samples with random size are important in many fields such as, biology, insurance and telephone engineering. Our objective in this Chapter is to study simple rank tests for the two-sample problem when the sample sizes are random. We will also develop tests for the change point problem, based on samples with random size. In fact the development of the two-sample results, in addition of being of interest in it's own, is essential in the development of the change point tests. To explain the connection between the change point problem and the general two-sample problem, let us consider both in the case when the sample size is fixed. Given a sequence of independent observations $X_1, \ldots, X_n$ , the null hypothesis in both problems is; $H_o : X_1, \ldots, X_n$ are independent identically distributed random variables. The at-most one change point alternative is ; $H_a$ : there exists $k, 1 \leq k < n$ such that $X_1, \ldots, X_k$ have a common distribution function $F(.)$, $X_{k+1}, \ldots, X_n$ have a common distribution function $G(.)$, where $F \neq G$ are assumed unknown. If the change point (position) $k$ is known, then the testing problem becomes a typical two-sample problem, based on the two independent samples $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$.

consequently a given test statistic for the two-sample problem for a given $k$ can be used to define a family of test statistics (indexed by $k$) which can be used to define test statistics for the corresponding at-most one change point problem. To explain this point, assume that; $T_{n,k} = T_n(X_1, \ldots, X_k; X_{k+1}, \ldots, X_n)$ is a test statistic for a two-sample problem based on the independent samples $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$. For the corresponding change point problem (i.e. when $k$ is unknown), we can define test statistics using appropriate functionals of the $T_{n,k}$ such as; $\max_{1 \leq k < n} T_{n,k} (\mid T_{n,k} \mid)$ or $\frac{1}{n} \sum_{k=1}^{n-1} T_{n,k}^2$.

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with continuous distribution function $F(.)$ and $\{N_n, n \geq 1\}$ be a sequence of nonnegative integer-valued random variables. In this part of the thesis we study tests based on samples of the form $N_n, X_1, \ldots, X_{N_n}$. Many researchers have studied properties and tests based on the so called modified empirical distribution function constructed from samples with random size. Allen and Beekman (1966) introduced a one sided Kac-type statistic similar to the one sided Kolmogorov statistic, obtained its asymptotic distribution and showed its consistency when $N_n$ is independent of the $X_i$'s. Csörgő, S. (1981) obtained strong approximations for the empirical Kac process when the sample size is independent of the random variables $X$ and follows the Poisson distribution. Mirzakhmedov and Tursunov (1992), were first to study the rate of convergence of the empirical process constructed from a sample of random size. The two-sample results we introduce here generalize the results of Aly *et al.* (1987) to the case of random sample size. Our work in this chapter is divided into two main parts. The first part (sections 2, 3 and 4) is devoted to exploring the asymptotic representations and distributions of the rank test in the two-sample problem. In section 5, we derive the asymptotic distribution of rank tests for the change point problem.

## 4.2 Definitions

Let $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ be two independent sequences of independent random variables. The random variables $X$ and $Y$ have distribution functions $F$ and $G$, respectively. Let $\{M_m, m \geq 1\}$ and $\{N_n, n \geq 1\}$ be two independent sequences of nonnegative integer-valued random variables. We will assume throughout this chapter that, there exist sequences of real numbers $a_n$, $b_n$, $a_m^*$ and $b_m^*$ such that $a_n \to 0$, $b_n \to 0$ as $n \to \infty$, $a_m^* \to 0$, $b_m^* \to 0$ as $m \to \infty$,

$$P(\mid \frac{N_n}{n} - \tau_1 \mid > a_n) \leq b_n \quad \text{and} \quad P(\mid \frac{M_m}{m} - \tau_2 \mid > a_m^*) \leq b_m^*, \tag{4.1}$$

where $0 < \tau_1, \tau_2 \leq 1$ are constants. The random variables $N_n$ (or $M_m$) satisfying (4.1) can be taken to be Binomial, Poisson and the number of renewals, (see Mirzakhmedov and Tursunov (1992)).

Let $\{X_i, 1 \leq i \leq M_m\}$ and $\{Y_j, 1 \leq j \leq N_n\}$ be two independent random size samples from the distributions $F$ and $G$ respectively. Define $F_{M_m}(.)$ and $G_{N_n}(.)$ as the empirical distribution functions of the samples $\{X_i, 1 \leq i \leq M_m\}$ and $\{Y_j, 1 \leq j \leq N_n\}$ respectively. Let $F_{N_n}^{-1}$ and $G_{M_m}^{-1}$ be the corresponding generalized inverses. For simplicity we will write $N := N_n$ and $M := M_m$. The P-P plot process for $X_1, \ldots, X_M$ and $Y_1, \ldots, Y_N$ is given by:

$$l^{*}_{m,n}(y) = (\frac{nm}{m+n})^{\frac{1}{2}}\{G_N F_M^{-1}(y) - GF^{-1}(y)\} \qquad ,0 \le y \le 1 \tag{4.2}$$

We shall assume throughout, that $0 < \lambda_* \le \frac{m}{n+m} \le 1 - \lambda_* < 1$, $\lambda_* \le \frac{1}{2}$.

Consider the problem of testing the hypotheses;

$$H_\circ : F = G \qquad \text{vs} \qquad H_1 : F \neq G, \tag{4.3}$$

where $F$ and $G$ are unknown continuous distribution functions . Note that under $H_\circ$ of (4.3), we have $F = G$ and the P-P plot process of (4.2) becomes

$$l_{m,n}(y) = \sqrt{\frac{nm}{m+n}}\,\{E_N V_M^{-1}(y) - y\}, \quad 0 \le y \le 1, \tag{4.4}$$

where $E_k :=$ EDF based on $G(Y_j)$, $j = 1, 2, \ldots, k$ and $V_l :=$ EDF based on $F(X_i)$, $i = 1, 2, \ldots, l$ are the Uniform-(0,1) empirical distribution functions and $V_l^{-1}(.)$ is the empirical quantile function based on $F(X_i)$, $i = 1, 2, \ldots, l$.

For $0 \le y \le 1$ we have from (4.4)

$$\begin{aligned} l_{m,n}(y) &= (\frac{nm}{m+n})^{1/2}\{E_N V_M^{-1}(y) - y\} \\ &= (\frac{nm}{m+n})^{1/2}\{(E_N V_M^{-1}(y) - V_M^{-1}(y)) - (y - V_M^{-1}(y))\} \\ &= (\frac{nm}{m+n})^{1/2}\{N^{-1/2}\Gamma_N(V_M^{-1}(y)) - M^{-1/2}\Upsilon_M(y)\}, \end{aligned} \tag{4.5}$$

where for $k \ge 2$ the uniform empirical and quantile processes $\Gamma_k(.)$ and $\Upsilon_k(.)$ are defined by:

$$\Gamma_k(t) = k^{1/2}\{E_k(t) - t\}, \qquad 0 \le t \le 1 \tag{4.6}$$

and

$$\Upsilon_k(t) = \begin{cases} k^{1/2}(t - V_k^{-1}(t)) & \text{for } t \in [\frac{1}{k+1}, \frac{k}{k+1}] \\ \\ 0 & \text{for } t \in [0, \frac{1}{k+1}) \bigcup (\frac{k}{k+1}, 1] \ . \end{cases} \qquad (4.7)$$

Next we prove some convergence results for the P-P plot processes of (4.4).

## 4.3 Convergence results for the two-sample problem

A Kiefer process $\mathcal{K}(s,t)$ is a mean zero Gaussian process with covariance function $E\mathcal{K}(s_1,t_1)\mathcal{K}(s_2,t_2) = (t_1 \wedge t_2)(s_1 \wedge s_2 - s_1 s_2)$. For the existence and properties of the Kiefer process we refer to Csörgő and Révész (1981). The next result is due to Kolmós, Major and Tusnády (1975). This Theorem will be used in this Chapter to obtain the convergence of the empirical processes involved in (4.9).

**Theorem A** Kolmós, Major and Tusnády (1975)

We can define a Kiefer process $\{\mathcal{K}(t,x),\, 0 \le t \le 1,\, x \ge 0\}$ such that

$$P\{\max_{1\le k\le n} \sup_{0\le t\le 1} \mid k^{\frac{1}{2}}\Gamma_k(t) - \mathcal{K}(t,k) \mid > (C_1 \log n + x)\log n\} \le C_2 \exp(-C_3 x),$$

for all $x > 0$, where $C_1$, $C_2$ and $C_3$ are positive constants and $\Gamma_k(.)$ is the uniform empirical process.

We also need the following result for the uniform empirical quantile process $\Upsilon_k(.)$, which is Theorem (3.2.4) in Csörgő and Horváth (1993).

**Theorem B**

We can define a Kiefer process $\{\mathcal{K}(t,x),\, 0 \le t \le 1,\, x \ge 0\}$ such that

$$\overline{\lim_{n\to\infty}} \sup_{0\le t\le 1} \mid n^{\frac{1}{2}}\Upsilon_n(t) - \mathcal{K}(t,n) \mid / \{n^{\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log_2 n)^{\frac{1}{4}}\} = 2^{-\frac{1}{4}} \quad \text{a.s.},$$

where $\log_2(.) = \log\log(.)$.

**Lemma** (4.1)

Let $\Gamma$ and $\Upsilon$ be defined by (4.6) and (4.7). Then, there exist two independent Kiefer processes $\mathcal{K}_1(.,.)$ and $\mathcal{K}_2(.,.)$ such that as $n \to \infty$ and $m \to \infty$ we have;

$$(i)\ \sup_{0 \le t \le 1} | N^{\frac{1}{2}}\Gamma_N(t) - \mathcal{K}_1(t,N) | \overset{P}{=} O(\log^2 n).$$

$$(ii)\ \sup_{0 \le t \le 1} | M^{\frac{1}{2}}\Upsilon_M(t) - \mathcal{K}_2(t,M) | / \sqrt{m} \overset{P}{=} o(1).$$

**Proof** of (i): Let $n_1 = n(\tau_1 - a_n)$, $n_2 = n(\tau_1 + a_n)$ and $\rho_n = \log^2 n$, where $a_n$ and $\tau_1$ are as in (4.1). Then by (4.1) we have as $n \to \infty$,

$$\begin{aligned}
& P(\sup_{0\le t\le 1} | N^{\frac{1}{2}}\Gamma_N(t) - \mathcal{K}_1(t,N) | / \rho_n > \lambda) \\
\le\ & P(\sup_{0\le t\le 1} | N^{\frac{1}{2}}\Gamma_N(t) - \mathcal{K}_1(t,N) | / \rho_n > \lambda, | \frac{N}{n} - \tau_1 | \le a_n) \\
& + P(| \frac{N}{n} - \tau_1 | > a_n) \\
\le\ & P(\sup_{0\le t\le 1} | N^{\frac{1}{2}}\Gamma_N(t) - \mathcal{K}_1(t,N) | / \rho_n > \lambda, n_1 \le N \le n_2) + b_n \\
\le\ & P(\sup_{0\le y\le n_2} \sup_{0\le t\le 1} | \sqrt{y}\,\Gamma_y(t) - \mathcal{K}_1(t,y) | > \lambda\rho_n) + b_n, \qquad \lambda > 0. \qquad (4.8)
\end{aligned}$$

Choosing $x = (\lambda - C_1)\log n$, in Theorem A we get

$$\begin{aligned}
P(\sup_{0\le y\le n_2} \sup_{0\le t\le 1} | \sqrt{y}\,\Gamma_y(t) - \mathcal{K}_1(t,y) | > \lambda\rho_n) \ &\le\ C_2 \exp\{-(\lambda - C_1)C_3 \log n\} \\
&=\ C_2 n^{-(\lambda - C_1)C_3} \to 0, \qquad (4.9)
\end{aligned}$$

for any $\lambda > C_1$.

By (4.8) and (4.9) we finish the proof of (i).

**Proof** of (ii):

Let $m_1 = m(\tau_2 - a_m^*)$, $m_2 = m(\tau_2 + a_m^*)$ and $a_m^*$ as in (4.1). Then by (4.1) we have for any $\epsilon > 0$

$$\begin{aligned}
& P(\sup_{0\le y\le 1} \mid M^{\frac{1}{2}}\Upsilon_M(y) - \mathcal{K}_2(y,M) \mid /\sqrt{m} > \epsilon) \\
\le\ & P(\sup_{0\le y\le 1} \mid M^{\frac{1}{2}}\Upsilon_M(y) - \mathcal{K}_2(y,M) \mid /\sqrt{m} > \epsilon, \mid \frac{M}{m} - \tau_2 \mid\le a_m^*) \\
& +P(\mid \frac{M}{m} - \tau_2 \mid > a_m^*) \\
\le\ & P(\sup_{0\le y\le 1} \mid M^{\frac{1}{2}}\Upsilon_M(y) - \mathcal{K}_2(y,M) \mid /\sqrt{m} > \epsilon, m_1 \le M \le m_2) + b_m^* \\
\le\ & P(\sup_{m_1\le k\le m_2} \sup_{0\le y\le 1} \mid k^{\frac{1}{2}}\Upsilon_k(y) - \mathcal{K}_2(y,k) \mid /\sqrt{m} > \epsilon) + b_m^*. \qquad (4.10)
\end{aligned}$$

By Theorem B, we have as $m \to \infty$

$$\begin{aligned}
& \sup_{m_1\le k\le m_2} \sup_{0\le y\le 1} \mid k^{\frac{1}{2}}\Upsilon_k(y) - \mathcal{K}_2(y,k) \mid /\sqrt{m} \\
\le\ & \sup_{m_1\le k\le m_2} \sup_{0\le y\le 1} \mid k^{\frac{1}{2}}\Upsilon_k(y) - \mathcal{K}_2(y,k) \mid /\sqrt{k}\,.\ \sup_{m_1\le k\le m_2} (\frac{k}{m})^{1/2} \\
\le\ & \sup_{k\ge m_1} \sup_{0\le y\le 1} \mid k^{\frac{1}{2}}\Upsilon_k(y) - \mathcal{K}_2(y,k) \mid /\sqrt{k}\ .\sqrt{\tau_2 + a_m^*} \\
\le\ & \overline{\lim_{k\to\infty}} \sup_{0\le y\le 1} \mid k^{\frac{1}{2}}\Upsilon_k(y) - \mathcal{K}_2(y,k) \mid /\sqrt{k} \\
=\ & o(1) \qquad \text{a.s.}. \qquad (4.11)
\end{aligned}$$

Hence by (4.10) and (4.11) we complete the proof of (ii).

In the following Theorem we obtain approximations for the P-P plot process.

**Theorem** (4.1)

Under $H_o$ of (4.3), there exist two independent Kiefer processes $\mathcal{K}_1(.,.)$ and

$\mathcal{K}_2(.,.)$ such that as $(n \wedge m) \to \infty$,

$$\begin{aligned}\mathcal{B}_{m,n} &= \sup_{0 \le y \le 1} \mid l_{m,n}(y) - \{(\frac{m}{m+n})^{\frac{1}{2}} \frac{n^{-\frac{1}{2}}}{\tau_1} \mathcal{K}_1(y, n\tau_1) - (\frac{n}{m+n})^{\frac{1}{2}} \frac{m^{-\frac{1}{2}}}{\tau_2} \\ &\quad .\mathcal{K}_2(y, m\tau_2)\} \mid \\ &\overset{\mathrm{P}}{=} o(1),\end{aligned}$$

where $l_{m,n}(.)$ is defined by (4.4).

**Proof** :

By (4.5) we have

$$\mathcal{B}_{m,n} \le \theta_{m,n,1} + \theta_{m,n,2}, \qquad \text{a.s.,} \tag{4.12}$$

where,

$$\theta_{m,n,1} = (\frac{m}{m+n})^{\frac{1}{2}} \sup_{0 \le y \le 1} \mid (\frac{n}{N}) N^{\frac{1}{2}} \Gamma_N(V_M^{-1}(y)) - \frac{1}{\tau_1} \mathcal{K}_1(y, n\tau_1) \mid / \sqrt{n} \tag{4.13}$$

and

$$\theta_{m,n,2} = (\frac{n}{m+n})^{\frac{1}{2}} \sup_{0 \le y \le 1} \mid (\frac{m}{M}) M^{\frac{1}{2}} \Upsilon_M(y) - \frac{1}{\tau_2} \mathcal{K}_2(y, m\tau_2) \mid / \sqrt{m}. \tag{4.14}$$

For $\theta_{m,n,1}$ of (4.13) we have

$$\theta_{m,n,1} \le \theta_{m,n,1}^{(1)} + \theta_{m,n,1}^{(2)} + \theta_{m,n,1}^{(3)} + \theta_{m,n,1}^{(4)}, \qquad \text{a.s. ,} \tag{4.15}$$

where,

$$\begin{aligned}\theta_{m,n,1}^{(1)} &= (\frac{m}{m+n})^{\frac{1}{2}} (\frac{N}{n})^{-1} \sup_{0 \le y \le 1} \mid N^{\frac{1}{2}} \Gamma_N(V_M^{-1}(y)) - \mathcal{K}_1(V_M^{-1}(y), N) \mid / \sqrt{n}, \\ \theta_{m,n,1}^{(2)} &= (\frac{m}{m+n})^{\frac{1}{2}} (\frac{N}{n})^{-1} \sup_{0 \le y \le 1} \mid \mathcal{K}_1(V_M^{-1}(y), N) - \mathcal{K}_1(y, N) \mid / \sqrt{n},\end{aligned}$$

$$\theta^{(3)}_{m,n,1} = (\frac{m}{m+n})^{\frac{1}{2}}(\frac{N}{n})^{-1} \sup_{0\le y\le 1} \mid \mathcal{K}_1(y,N) - \mathcal{K}_1(y,n\tau_1) \mid /\sqrt{n}$$

and

$$\theta^{(4)}_{m,n,1} = (\frac{m}{m+n})^{\frac{1}{2}} \sup_{0\le y\le 1} \mid (\frac{n}{N})\mathcal{K}_1(y,n\tau_1) - \frac{1}{\tau_1}\mathcal{K}_1(y,n\tau_1) \mid /\sqrt{n}.$$

From (4.12) and (4.15) we have for any $\epsilon > 0$,

$$\begin{aligned} P(\mathcal{B}_{m,n} > \epsilon) &\le P(\theta_{m,n,1} > \epsilon/2) + P(\theta_{m,n,2} > \epsilon/2) \\ &\le P(\theta^{(1)}_{m,n,1} > \epsilon/8) + P(\theta^{(2)}_{m,n,1} > \epsilon/8) + P(\theta^{(3)}_{m,n,1} > \epsilon/8) \\ &\quad + P(\theta^{(4)}_{m,n,1} > \epsilon/8) + P(\theta_{m,n,2} > \epsilon/2). \end{aligned} \tag{4.16}$$

Next we will handle each of the probabilities in (4.16) separately.

By Lemma (4.1) we get as $(m \wedge n) \to \infty$

$$\begin{aligned} P(\theta^{(1)}_{m,n,1} > \epsilon/8) &\le P((\frac{N}{n})^{-1} \sup_{0\le x\le 1} \mid N^{\frac{1}{2}}\Gamma_N(x) - \mathcal{K}_1(x,N) \mid /\sqrt{n} > \epsilon/8) \\ &= o(1). \end{aligned} \tag{4.17}$$

Using (4.1) we obtain as $(m \wedge n) \to \infty$

$$\begin{aligned} P(\theta^{(4)}_{m,n,1} > \epsilon/8) &\le P(\theta^{(4)}_{m,n,1} > \epsilon/8, \mid \frac{N}{n} - \tau_1 \mid \le a_n) + b_n \\ &\le P(n^{-1/2} \sup_{0\le y\le 1} \mid \mathcal{K}_1(y,n\tau_1) \mid \{\mid (\frac{N}{n})^{-1} - \tau_1^{-1} \mid\} > \epsilon/8, \\ &\qquad \mid \frac{N}{n} - \tau_1 \mid \le a_n) + b_n \\ &\le P(n^{-1/2} \sup_{0\le y\le 1} \mid \mathcal{K}_1(y,n\tau_1) \mid \{\mid \frac{\tau_1 - (\frac{N}{n})}{\tau_1(\frac{N}{n})} \mid\} > \epsilon/8, \\ &\qquad \mid \frac{N}{n} - \tau_1 \mid \le a_n) + b_n \end{aligned}$$

$$\begin{aligned}
&\leq \quad P(n^{-1/2} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(y, n\tau_1) \mid \{\frac{a_n}{\tau_1(\tau_1 - a_n)}\} > \epsilon/8) + b_n \\
&= \quad o(1). \qquad (4.18)
\end{aligned}$$

By (6) of Mirzakhmedov and Tursunov (1992) and (4.1) we have as $(m \wedge n) \to \infty$

$$\begin{aligned}
P & \quad (\theta^{(3)}_{m,n,1} > \epsilon/8) \\
\leq & \quad P(\theta^{(3)}_{m,n,1} > \epsilon/8, \mid \frac{N}{n} - \tau_1 \mid \leq a_n) + b_n \\
\leq & \quad P(\frac{n^{-1/2}}{(\tau_1 - a_n)} \sup_{0\leq z\leq na_n} \sup_{0\leq x\leq n\tau_1} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(y, x+z) - \mathcal{K}_1(y, x) \mid > \epsilon/8) + b_n \\
\leq & \quad P(\frac{2}{(\tau_1 - a_n)} \sup_{0\leq z\leq na_n} \sup_{0\leq x\leq n} \sup_{0\leq y\leq 1} \mid \frac{W(y, x+z)}{\sqrt{n}} - \frac{W(y, x)}{\sqrt{n}} \mid > \epsilon/8) + b_n \\
\leq & \quad P(\frac{2}{(\tau_1 - a_n)} \sup_{0\leq z\leq a_n} \sup_{0\leq x\leq 1} \sup_{0\leq y\leq 1} \mid W^\circ(y, x+z) - W^\circ(y, x) \mid > \epsilon/8) + b_n \\
= & \quad o(1), \qquad (4.19)
\end{aligned}$$

where $W(x,y)$ and $W^\circ(x,y)$, $(x,y) \in [0,1] \times [0,1]$ are standard Wiener fields.

Let $\eta > 0$ and $h_m = \eta.(\frac{\log m}{m})^{1/2}$, then by the convergence of the empirical quantile process, (4.1) and Lemma (1.11.3) of Csörgő and Révész (1981) we have as $(m \wedge n) \to \infty$

$$\begin{aligned}
P & \quad (\theta^{(2)}_{m,n,1} > \epsilon/8) \\
\leq & \quad P((\frac{N}{n})^{-1} n^{-\frac{1}{2}} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(V_M^{-1}(y), N) - \mathcal{K}_1(y, N) \mid > \frac{\epsilon}{8}, \mid \frac{N}{n} - \tau_1 \mid \leq a_n) + b_n \\
\leq & \quad P(\frac{n^{-\frac{1}{2}}}{(\tau_1 - a_n)} \sup_{0\leq k\leq n_2} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(V_M^{-1}(y), k) - \mathcal{K}_1(y, k) \mid > \frac{\epsilon}{8}) + b_n \\
\leq & \quad P(\frac{n^{-\frac{1}{2}}}{(\tau_1 - a_n)} \sup_{0\leq k\leq n_2} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(V_M^{-1}(y), k) - \mathcal{K}_1(y, k) \mid > \frac{\epsilon}{8}, \\
& \quad \sup_{0\leq y\leq 1} M^{\frac{1}{2}} \mid V_M^{-1}(y) - y \mid \leq (\log m)^{\frac{1}{2}}) + P(\sup_{0\leq y\leq 1} \mid \Upsilon_M(y) \mid > (\log m)^{\frac{1}{2}}) + b_n
\end{aligned}$$

$$\begin{aligned}
&\leq P\Big(\frac{n^{-\frac{1}{2}}}{(\tau_1 - a_n)} \sup_{0\leq k\leq n_2} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(V_M^{-1}(y), k) - \mathcal{K}_1(y,k) \mid > \frac{\epsilon}{8}, (\frac{M}{m})^{1/2}. \\
&\quad \sup_{0\leq y\leq 1} \mid V_M^{-1}(y) - y \mid \leq (\frac{\log m}{m})^{\frac{1}{2}}\Big) + P(\sup_{0\leq y\leq 1} \mid \Upsilon_M(y) \mid > (\log m)^{\frac{1}{2}}) + b_n \\
&\leq P\Big(\frac{n^{-\frac{1}{2}}}{(\tau_1 - a_n)} \sup_{0\leq x\leq h_m} \sup_{0\leq k\leq n_2} \sup_{0\leq y\leq 1} \mid \mathcal{K}_1(y+x, k) - \mathcal{K}_1(y,k) \mid > \frac{\epsilon}{8}\Big) + b_m^{*} \\
&\quad + P(\sup_{0\leq y\leq 1} \mid \Upsilon_M(y) \mid > (\log m)^{\frac{1}{2}}) + b_n \\
&= o(1). \qquad (4.20)
\end{aligned}$$

From (4.16)-(4.20) we get for any $\epsilon > 0$,

$$P(\theta_{m,n,1} > \frac{\epsilon}{2}) = o(1), \qquad \text{as } (m \wedge n) \to \infty. \qquad (4.21)$$

As to $\theta_{m,n,2}$ of (4.14) we have a.s.

$$\theta_{m,n,2} \leq \theta_{m,n,2}^{(1)} + \theta_{m,n,2}^{(2)} + \theta_{m,n,2}^{(3)}\ , \qquad (4.22)$$

where,

$$\theta_{m,n,2}^{(1)} = (\frac{m}{m+n})^{\frac{1}{2}} (\frac{M}{m})^{-1} \sup_{0\leq t\leq 1} \mid M^{\frac{1}{2}} \Upsilon_M(t) - \mathcal{K}_2(t, M) \mid / \sqrt{m},$$

$$\theta_{m,n,2}^{(2)} = (\frac{m}{m+n})^{\frac{1}{2}} (\frac{M}{m})^{-1} \sup_{0\leq t\leq 1} \mid \mathcal{K}_2(t, M) - \mathcal{K}_2(t, m\tau_2) \mid / \sqrt{m},$$

and

$$\theta_{m,n,2}^{(3)} = (\frac{m}{m+n})^{\frac{1}{2}} \sup_{0\leq t\leq 1} \mid \mathcal{K}_2(t, m\tau_2) \mid / \sqrt{m}.\{\mid (\frac{M}{m})^{-1} - \tau_2^{-1} \mid\}.$$

Hence,

$$P(\theta_{m,n,2} > \frac{\epsilon}{2}) \leq P(\theta_{m,n,2}^{(1)} > \frac{\epsilon}{6}) + P(\theta_{m,n,2}^{(2)} > \frac{\epsilon}{6}) + P(\theta_{m,n,2}^{(3)} > \frac{\epsilon}{6}). \qquad (4.23)$$

Replacing $n$ by $m$ in (4.18) and (4.19) we get as $(m \wedge n) \to \infty$

$$P(\theta^{(3)}_{m,n,2} > \frac{\epsilon}{6}) = o(1) \tag{4.24}$$

and

$$P(\theta^{(2)}_{m,n,2} > \frac{\epsilon}{6}) = o(1). \tag{4.25}$$

Using (ii) of Lemma (4.1) and (4.1) we have as $(m \wedge n) \to \infty$

$$\begin{aligned} P(\theta^{(1)}_{m,n,2} > \frac{\epsilon}{6}) &\leq P(\theta^{(1)}_{m,n,2} > \frac{\epsilon}{6}, \mid \frac{M}{m} - \tau_2 \mid \leq a^*_m) + b^*_m \\ &\leq P(\frac{m^{-\frac{1}{2}}}{(\tau_2 - a^*_m)} \sup_{0 \leq t \leq 1} \mid M^{\frac{1}{2}} \Upsilon_M(t) - \mathcal{K}_2(t, M) \mid > \frac{\epsilon}{6}) + b^*_m \\ &= o(1). \end{aligned} \tag{4.26}$$

From (4.23)-(4.26) we get for any $\epsilon > 0$,

$$P(\theta_{m,n,2} > \frac{\epsilon}{2}) = o(1), \qquad \text{as } (m \wedge n) \to \infty. \tag{4.27}$$

By (4.16), (4.21) and (4.27) we finish the proof of Theorem (4.1).

Define; for $0 \leq y \leq 1$

$$H^*_{m,n}(y) = \frac{M}{M+N} F_M(y) + \frac{N}{M+N} G_N(y), \tag{4.28}$$

$$S^*_{m,n}(y) = (\frac{nm}{m+n})^{\frac{1}{2}} \{H^*_{m,n}(F^{-1}_M(y)) - y\} \tag{4.29}$$

where $S^*_{m,n}(.)$, are called the "Quantile Rank" process (cf. Parzen (1983) and Aly *et al.* (1987)).

We note that under $H_o$ of (4.3) and as in (4.4), the process $S^{*}_{m,n}(.)$, of (4.29) become

$$
\begin{aligned}
S_{m,n}(y) &= (\frac{nm}{m+n})^{\frac{1}{2}}\{H_{m,n}(V_M^{-1}(y)) - y\} \\
&= (\frac{nm}{m+n})^{\frac{1}{2}}\{\frac{M}{M+N}V_M(V_M^{-1}(y)) + \frac{N}{M+N}E_N(V_M^{-1}(y)) - y\} \\
&= (\frac{nm}{m+n})^{\frac{1}{2}}\{\frac{N}{M+N}(E_N(V_M^{-1}(y)) - y) + \frac{M}{M+N}(V_M(V_M^{-1}(y)) - y)\} \\
&= \frac{N}{M+N}l_{m,n}(y) + (\frac{nm}{m+n})^{\frac{1}{2}}\frac{M}{M+N}(V_M(V_M^{-1}(y)) - y), \qquad (4.30)
\end{aligned}
$$

where $0 \leq y \leq 1$ and $l_{m,n}(.)$, is defined by (4.5).

Now for the second term of $S_{m,n}(.)$ in (4.30) we have by (4.1) and as $\lambda \to \infty$, $(m \wedge n) \to \infty$

$$
\begin{aligned}
& P\{(\frac{nm}{m+n})^{\frac{1}{2}}(\frac{M}{M+N})\, m^{\frac{1}{2}} \sup_{0\leq y\leq 1} \mid V_M(V_M^{-1}(y)) - y \mid > \lambda\} \\
\leq\ & P\{(\frac{m}{m+n})^{\frac{1}{2}}(\frac{M}{M+N})m \sup_{0\leq y\leq 1} \mid V_M(V_M^{-1}(y)) - y \mid > \lambda\} \\
\leq\ & P\{m \sup_{0\leq y\leq 1} \mid V_M(V_M^{-1}(y)) - y \mid > \lambda\} \\
\leq\ & P\{\frac{m}{M} > \lambda\} = P\{\frac{M}{m} < \frac{1}{\lambda}\} \\
=\ & o(1),
\end{aligned}
$$

and hence

$$
(\frac{nm}{m+n})^{\frac{1}{2}}(\frac{M}{M+N})\{V_M(V_M^{-1}(y)) - y\} \overset{p}{=} O(m^{-\frac{1}{2}}). \qquad (4.31)
$$

From (4.30) and (4.31) we have

$$
S_{m,n}(y) = (\frac{N}{M+N})l_{m,n}(y) + O_p(m^{-\frac{1}{2}}), \qquad 0 \leq y \leq 1. \qquad (4.32)
$$

Under $H_\circ$ of (4.3), we define

$$D^{-1}_{m,n} = (H_{m,n}V_M^{-1})^{-1}(y), \qquad 0 \le y \le 1 \tag{4.33}$$

and

$$R_{m,n}(y) = (\frac{nm}{m+n})^{1/2}\{D^{-1}_{m,n}(y) - y\} \qquad 0 \le y \le 1 \tag{4.34}$$

where $H_{m,n}(.)$ as in (4.30). The process $R_{m,n}(.)$, is called "Empirical Rank" process (cf. Parzen (1983) and Aly *et al.* (1987)).

Using the definitions of the processes in (4.30) and (4.34), we obtain the following relationship.

$$\begin{aligned} R_{m,n}(y) &= (\frac{nm}{m+n})^{1/2}\{(H_{m,n}V_M^{-1})^{-1}(y) - y\}, \qquad 0 \le y \le 1 \\ &= (\frac{nm}{m+n})^{1/2}\{(H_{m,n}V_M^{-1})^{-1}(y) - H_{m,n}V_M^{-1}(H_{m,n}V_M^{-1})^{-1}(y)\} \\ &\quad +(\frac{nm}{m+n})^{1/2}\{H_{m,n}V_M^{-1}(H_{m,n}V_M^{-1})^{-1}(y) - y\} \\ &= -S_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) + I_{m,n}(y), \end{aligned} \tag{4.35}$$

where,

$$I_{m,n}(y) = (\frac{nm}{m+n})^{1/2}\{H_{m,n}V_M^{-1}(H_{m,n}V_M^{-1})^{-1}(y) - y\}, \quad 0 \le y \le 1, \tag{4.36}$$

which we will show later that it vanishes in probability.

**Theorem** (4.2)

Let $\mathcal{K}_1(.,.)$, $\mathcal{K}_2(.,.)$, be the Kiefer processes of Theorem (4.1). Then under $H_\circ$ of (4.3) we have

$$\mathcal{C}_{m,n} := \sup_{0\le y\le 1} \mid S_{m,n}(y) - \Delta^*_{m,n}(y) \mid \overset{P}{=} o(1)$$

where $S_{m,n}(.)$, is defined by (4.29),(under $H_o$ of (4.3)) and

$$\Delta^{\bullet}_{m,n}(y) \;=\; \frac{n\tau_1}{(m\tau_2+n\tau_1)}\{(\frac{m}{m+n})^{\frac{1}{2}}\frac{\mathcal{K}_1(y,n\tau_1)}{\tau_1\sqrt{n}} - (\frac{n}{m+n})^{\frac{1}{2}}\frac{\mathcal{K}_2(y,m\tau_2)}{\tau_2\sqrt{m}}\}, \tag{4.37}$$

**Proof**

Using Theorem (4.1) and (4.32) we obtain as $(m \wedge n) \to \infty$

$$\begin{aligned} \mathcal{C}_{m,n} \;&\leq\; \sup_{0\leq y\leq 1} \mid \frac{N}{M+N} l_{m,n}(y) - \Delta^{\bullet}_{m,n}(y) \mid + O_p(m^{-\frac{1}{2}}) \\ &\leq\; \frac{N}{M+N}\sup_{0\leq y\leq 1} \mid l_{m,n}(y) - (\frac{n\tau_1}{m\tau_2+n\tau_1})^{-1}\Delta^{\bullet}_{m,n}(y) \mid \\ &\quad + \sup_{0\leq y\leq 1} \mid \Delta^{\bullet}_{m,n}(y)\{(\frac{N}{M+N})(\frac{n\tau_1}{m\tau_2+n\tau_1})^{-1} - 1\} \mid + O_p(m^{-\frac{1}{2}}) \\ &=\; o_p(1) + \sup_{0\leq y\leq 1} \mid \Delta^{\bullet}_{m,n}(y)\{(\frac{N}{M+N})(\frac{n\tau_1}{m\tau_2+n\tau_1})^{-1} - 1\} \mid + o_p(1). \end{aligned} \tag{4.38}$$

By the definition of $\Delta^{\bullet}_{m,n}(.)$ in (4.37) we have for any $\epsilon > 0$

$$\begin{aligned} P\; &(\sup_{0\leq y\leq 1} \mid \Delta^{\bullet}_{m,n}(y)\{(\frac{N}{M+N})(\frac{n\tau_1}{m\tau_2+n\tau_1})^{-1} - 1\} \mid > \epsilon) \\ \leq\; &P((\frac{n^{-\frac{1}{2}}}{\tau_1})(\frac{m}{m+n})^{\frac{1}{2}}\{\sup_{0\leq y\leq 1} \mid \mathcal{K}_1(y,n\tau_1) \mid\}.\mid \frac{N}{M+N} - \frac{n\tau_1}{m\tau_2+n\tau_1} \mid > \frac{\epsilon}{2}) \\ &+P((\frac{m^{-\frac{1}{2}}}{\tau_2})(\frac{n}{m+n})^{\frac{1}{2}}\{\sup_{0\leq y\leq 1} \mid \mathcal{K}_2(y,m\tau_2) \mid\}.\mid \frac{N}{M+N} - \frac{n\tau_1}{m\tau_2+n\tau_1} \mid > \frac{\epsilon}{2}). \end{aligned} \tag{4.39}$$

Hence to show that (4.38) is $o_p(1)$ as $(m \wedge n) \to \infty$, it is enough to show that;

$$P((\frac{n^{-\frac{1}{2}}}{\tau_1})(\frac{m}{m+n})^{\frac{1}{2}}\{\sup_{0\leq y\leq 1} \mid \mathcal{K}_1(y,n\tau_1) \mid\}.\mid \frac{N}{M+N} - \frac{n\tau_1}{m\tau_2+n\tau_1} \mid > \frac{\epsilon}{2}) = o(1). \tag{4.40}$$

Conditional on the events $\{| \frac{N}{n} - \tau_1 | \leq a_n\}$ and $\{| \frac{M}{m} - \tau_2 | \leq a^*_m\}$ we have

$$\begin{aligned} \left| \frac{N}{M+N} - \frac{n\tau_1}{m\tau_2 + n\tau_1} \right| &\leq \left| \frac{m\tau_2 N - n\tau_1 M}{(M+N)(m\tau_2 + n\tau_1)} \right| \\ &\leq \frac{m\, n\tau_1\tau_2}{N\,(m\tau_2 + n\tau_1)} \left| (\frac{N}{n\tau_1}) - (\frac{M}{m\tau_2}) \right| \\ &\leq (\tau_1)(\frac{1}{(N/n)})\{| \frac{N}{n\tau_1} - 1 | + | \frac{M}{m\tau_2} - 1 |\} \\ &\leq (\tau_2)^{-1}(\frac{\tau_2 a_n + \tau_1 a^*_m}{\tau_1 - a_n}) \to 0 \qquad \text{as } (m \wedge n) \to \infty. \end{aligned} \tag{4.41}$$

We also notice that, for all $n \geq 1$ and $\tau_1 > 0$,

$$\sup_{0 \leq y \leq 1} | \mathcal{K}_1(y, n\tau_1)/\sqrt{n} \overset{d}{=} \sup_{0 \leq y \leq 1} | B(y) | \tau_1^{\frac{1}{2}} \overset{P}{=} O(1), \tag{4.42}$$

where $B(.)$ is a standard Brownian bridge. By (4.1), (4.41) and (4.42) we can prove (4.40).

Hence the proof of Theorem (4.2) is complete.

**Corollary** (4.1)

Let $\Delta^*_{m,n}(.)$, be defined by (4.37). Then, under $H_\circ$ of (4.3) we have as $(m \wedge n) \to \infty$

$$\sup_{0 \leq y \leq 1} | S_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^*_{m,n}(y) | \overset{P}{=} o(1),$$

where $S_{m,n}(.)$, is as defined by (4.30).

**Proof**

By Theorem (4.2) we have as $(m \wedge n) \to \infty$

$$\begin{aligned}
& \sup_{0 \le y \le 1} \mid S_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^{*}_{m,n}(y) \mid \\
\le\ & \sup_{0 \le y \le 1} \mid S_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^{*}_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) \mid \\
& + \sup_{0 \le y \le 1} \mid \Delta^{*}_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^{*}_{m,n}(y) \mid \\
\le\ & \sup_{0 \le x \le 1} \mid S_{m,n}(x) - \Delta^{*}_{m,n}(x) \mid \\
& + \sup_{0 \le y \le 1} \mid \Delta^{*}_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^{*}_{m,n}(y) \mid \\
=\ & o_p(1) + \sup_{0 \le y \le 1} \mid \Delta^{*}_{m,n}((H_{m,n}V_M^{-1})^{-1}(y)) - \Delta^{*}_{m,n}(y) \mid . \qquad (4.43)
\end{aligned}$$

By Horváth (1984)'s Lemma and the convergence of the process $S_{m,n}(.)$, we have

$$\begin{aligned}
\sup_{0 \le y \le 1} \mid (H_{m,n}V_M^{-1})^{-1}(y) - y \mid \ & \le \ \sup_{0 \le y \le 1} \mid H_{m,n}V_M^{-1}(y) - y \mid \\
& = \ (\frac{nm}{m+n})^{-\frac{1}{2}} \sup_{0 \le y \le 1} \mid S_{m,n}(y) \mid \\
& \overset{P}{=} \ O((m \wedge n)^{-\frac{1}{2}}). \qquad (4.44)
\end{aligned}$$

Let $\hat{y} = (H_{m,n}V_M^{-1})^{-1}(y)$, $0 \le y \le 1$ then by the definition of the process $\Delta^{*}_{m,n}(.)$, in (4.37) we have ;

$$\begin{aligned}
\sup_{0 \le y \le 1} \mid \Delta^{*}_{m,n}(\hat{y}) - \Delta^{*}_{m,n}(y) \mid \ & \le \ \sup_{0 \le y \le 1} \mid \mathcal{K}_1(\hat{y}, n\tau_1) - \mathcal{K}_1(y, n\tau_1) \mid /(\tau_1\sqrt{n}) \\
& + \sup_{0 \le y \le 1} \mid \mathcal{K}_2(\hat{y}, m\tau_2) - \mathcal{K}_2(y, m\tau_2) \mid /(\tau_2\sqrt{m}).
\end{aligned} \qquad (4.45)$$

Now let $h_{m,n} = c(\frac{\log(m \wedge n)}{(m \wedge n)})^{\frac{1}{2}}$, $c > 0$ , then for any $\epsilon > 0$ we have by (4.44) and (1.4.1) of Csörgő and Révész (1981),

$$P(\sup_{0 \le y \le 1} \mid \mathcal{K}_1(\hat{y}, n\tau_1) - \mathcal{K}_1(y, n\tau_1) \mid /(\tau_1\sqrt{n}) > \epsilon)$$

$$
\begin{aligned}
&= P(\sup_{0\le y\le 1} \mid B_1(\hat{y}) - B_1(y) \mid /(\sqrt{\tau_1}) > \epsilon) \\
&\le P(\sup_{0\le y\le 1} \mid B_1(\hat{y}) - B_1(y) \mid /(\sqrt{\tau_1}) > \epsilon, \sup_{0\le y\le 1} \mid \hat{y} - y \mid \le h_{m,n}) \\
&\quad + P(\sup_{0\le y\le 1} \mid \hat{y} - y \mid > h_{m,n}) \\
&\le P(\sup_{0\le x\le h_{m,n}} \sup_{0\le y\le 1} \mid \frac{B_1(y+x) - B_1(y)}{(x\log x^{-1})^{\frac{1}{2}}} \mid . \sup_{0\le x\le h_{m,n}} (x\log x^{-1})^{\frac{1}{2}} > \epsilon) \\
&\quad + P((m\wedge n)^{\frac{1}{2}}. \sup_{0\le y\le 1} \mid \hat{y} - y \mid > c(\log(m\wedge n))^{\frac{1}{2}}) \\
&= o(1), \qquad \text{as } (m\wedge n) \to \infty, \qquad (4.46)
\end{aligned}
$$

where $B_1(.)$ is a standard Brownian bridge.

Combining (4.43), (4.45) and (4.46) we complete the proof of the Corollary.

We now return to the reminder term in (4.35), i.e. to (4.36) and show that it is $o_p(1)$.

**Lemma** (4.2)

Let $I_{m,n}(.)$ be defined by (4.36), then we have as $(m\wedge n) \to \infty$

$$
\mathcal{I}_{m,n} = \sup_{0\le y\le 1} \mid I_{m,n}(y) \mid \overset{\mathrm{P}}{=} o(1),
$$

**Proof**

Let $\epsilon > 0$, then by (4.1) and (4.28) of Aly *et al.* (1987) , we get

$$
\begin{aligned}
P\ &(\mathcal{I}_{m,n} > \epsilon) \\
&\le P(\mathcal{I}_{m,n} > \epsilon, \mid \frac{M}{m} - \tau_2 \mid \le a^*_m, \mid \frac{N}{n} - \tau_1 \mid \le a_n) + b_n + b^*_m \\
&\le P(\sup_{n_1\le k\le n_2} \sup_{m_1\le l\le m_2} \sup_{0\le y\le 1} (\frac{nm}{m+n})^{\frac{1}{2}} \mid \tilde{H}_{m,n}V_k^{-1}(\tilde{H}_{m,n}V_k^{-1})^{-1}(y) - y \mid > \epsilon)
\end{aligned}
$$

$$
\begin{aligned}
& +b_n + b^*_m \\
\leq\ & P\Big( \sup_{n_1 \leq k \leq n_2} \sup_{m_1 \leq l \leq m_2} \sup_{0 \leq y \leq 1 - \frac{1}{k+l}} \sup_{0 \leq x \leq \frac{1}{k+l}} \mid \tilde{S}_{m,n}(y+x) - \tilde{S}_{m,n}(y) \mid > \epsilon \Big) \\
& +b_n + b^*_m, \qquad (4.47)
\end{aligned}
$$

where $\tilde{H}_{m,n}(.)$ and $\tilde{S}_{m,n}(.)$ are the processes defined in (4.30) when $M = l$ and $N = k$. Hence by Theorem (4.2) and (4.45)-(4.47) we have for any $\epsilon > 0$,

$$
P(\mathcal{I}_{m,n} > \epsilon) = o(1), \qquad \text{as} \, (m \wedge n) \to \infty.
$$

**Theorem** (4.3)

Let $R_{m,n}(.)$, be defined by (4.34). Then, under $H_\circ$ of (4.3) we have as $(m \wedge n) \to \infty$

$$
\sup_{0 \leq y \leq 1} \mid R_{m,n}(y) - \Delta_{m,n}(y) \mid \overset{\mathrm{P}}{=} o(1), \qquad ,
$$

where $\Delta_{m,n}(.) = -\Delta^*_{m,n}(.)$, as in (4.37).

The proof of this Theorem follows from (4.35), Corollary (4.1) and Lemma (4.2).

In the following we will show that the result of Theorem (4.3) holds true in the weighted sup-norm with a square integrable weight function $q(t)$, $0 < t < 1$.

Let $Q$ be the class of positive functions $q$ on $(0, 1)$, i.e. $\inf_{\theta \leq t \leq 1-\theta} q(t) > 0$ for all $0 < \theta < \frac{1}{2}$, non-decreasing in a neighbourhood of 0 and non-increasing in a neighbourhood of 1 and for which $\int_0^1 q^{-2}(t)\mathrm{d}t < \infty$, holds.

**Theorem** (4.4)

Let $q \in Q$, then we have as $(m \wedge n) \to \infty$

$$\sup_{0<y<1} \mid R_{m,n}(y) - \Delta_{m,n}(y) \mid /q(y) \overset{\mathrm{P}}{=} o(1),$$

where $R_{m,n}(.)$ and $\Delta_{m,n}(.)$, are as in Theorem (4.3) above.

**Proof**

For any $0 < \theta < \frac{1}{2}$, $\inf_{\theta \leq t \leq 1-\theta} q(t) > 0$ and hence, by Theorem (4.3) as $(m \wedge n) \to \infty$

$$\sup_{\theta \leq y \leq 1-\theta} \mid R_{m,n}(y) - \Delta_{m,n}(y) \mid /q(y) \overset{\mathrm{P}}{=} o(1). \tag{4.48}$$

By (4.48) and the symmetry over the intervals $0 < y < \theta$ and $1 - \theta < y < 1$, it is sufficient to show that for any $\epsilon > 0$;

$$\lim_{\theta \to 0} \limsup_{(m \wedge n) \to \infty} P( \sup_{0<y\leq\theta} \mid \Delta_{m,n}(y) \mid /q(y) > \epsilon) = 0, \tag{4.49}$$

and

$$\lim_{\theta \to 0} \limsup_{(m \wedge n) \to \infty} P( \sup_{0<y\leq\theta} \mid R_{m,n}(y) \mid /q(y) > \epsilon) = 0. \tag{4.50}$$

By the definition of the Gaussian process $\Delta^{*}_{m,n}(.) = -\Delta_{m,n}(.)$ in (4.37), the statement of (4.49) will be verified if we show that for any $\epsilon > 0$,

$$\lim_{\theta \to 0} P( \sup_{0<y\leq\theta} n^{-\frac{1}{2}} \mid \mathcal{K}(y, n\tau) \mid /q(y) > \epsilon) = 0, \tag{4.51}$$

for any Kiefer process, $\mathcal{K}(.,.)$, and all $n \geq 1$. But this is true since for every $n \geq 1$,

$$\begin{aligned} P( \sup_{0<y\leq\theta} n^{-\frac{1}{2}} \mid \mathcal{K}(y, n\tau) \mid /q(y) > \epsilon) &= P( \sup_{0<y\leq\theta} \mid \mathcal{K}^{\circ}(y, \tau) \mid /q(y) > \epsilon) \\ &= P(\sqrt{\tau} \sup_{0<y\leq\theta} \frac{\mid B^{\circ}(y) \mid}{q(y)} > \epsilon), \end{aligned} \tag{4.52}$$

where $\mathcal{K}^\circ(.,.)$ is a Kiefer process and $B^\circ(.)$ is a Brownian bridge. Hence by (4.51), (4.52) and (2.5) of Aly *et al.* (1987) we get

$$\lim_{\theta\to 0}\limsup_{(m\wedge n)\to\infty} P(\sup_{0\le y\le\theta} \mid \Delta_{m,n}(y) \mid /q(y) > \epsilon) = 0. \tag{4.53}$$

Now, to verify (4.50), we use Aly *et al.* (1987)'s representation for the process $R_{m,n}(.)$, and hence it is enough to show that

$$\lim_{\theta\to 0}\limsup_{(m\wedge n)\to\infty} P(\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon) = 0, \tag{4.54}$$

holds for any $\epsilon > 0$.

To prove (4.54) (and hence (4.50)) we proceed as follows. Let $\tilde{H}_{m,n}(.)$ be the process $H_{m,n}(.)$ of (4.30) when $M = k$ and $N = l$. Then by (4.1) we have for any $\epsilon > 0$;

$$\begin{aligned}
P & \ (\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon) \\
\le & \ P(\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon, \mid \frac{M}{m} - \tau_2 \mid \le a^*_m) + P(\mid \frac{M}{m} - \tau_2 \mid > a^*_m) \\
\le & \ P(\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon, \mid \frac{M}{m} - \tau_2 \mid \le a^*_m, \mid \frac{N}{n} - \tau_1 \mid \le a_n) \\
& \ +P(\mid \frac{N}{n} - \tau_1 \mid > a_n) + P(\mid \frac{M}{m} - \tau_2 \mid > a^*_m) \\
\le & \ P(\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon, m(\tau_2 - a^*_m) \le M \le m(\tau_2 + a^*_m), \\
& \ n(\tau_1 - a_n) \le N \le n(\tau_1 + a_n)) + b_n + b^*_m \\
\le & \ P(\sup_{0<y\le\theta} \mid \Gamma_M(H^{-1}_{m,n}(y)) \mid /q(y) > \epsilon, m_1 \le M \le m_2, n_1 \le N \le n_2) \\
& \ +b_n + b^*_m \\
\le & \ P(\max_{m_1\le k\le m_2}\ \max_{n_1\le l\le n_2}\ \sup_{0<y\le\theta} \mid \Gamma_k(\tilde{H}^{-1}_{m,n}(y)) \mid /q(y) > \epsilon) + b_n + b^*_m \\
= & \ g_{1,m,n} + g_{2,m,n} + b_n + b^*_m,
\end{aligned} \tag{4.55}$$

where,

$$g_{1,m,n} = P(\max_{m_1 \le k \le m_2} \max_{n_1 \le l \le n_2} \sup_{0<y<\frac{1}{k+l+1}} \mid \Gamma_k(\tilde{H}_{m,n}^{-1}(y)) \mid / q(y) > \epsilon/2), \quad (4.56)$$

$$g_{2,m,n} = P(\max_{m_1 \le k \le m_2} \max_{n_1 \le l \le n_2} \sup_{\frac{1}{k+l+1} \le y \le \theta} \mid \Gamma_k(\tilde{H}_{m,n}^{-1}(y)) \mid / q(y) > \epsilon/2). \quad (4.57)$$

Let $e_{m,n} = O_{\text{a.s.}}((m+n)^{-\frac{1}{2}}(\log\log(m+n))^{\frac{1}{2}}) = o_{\text{a.s.}}(1)$ as $(m \wedge n) \to \infty$. Then for $g_{2,m,n}$ of (4.57) we have by the LIL of the quantile process (see Shorack and Wellner (1986)),

$$\begin{aligned} & g_{2,m,n} \\ \le\ & P\{\max_{m_1 \le k \le m_2} \max_{n_1 \le l \le n_2} \sup_{\frac{1}{k+l+1} \le y \le \theta} \mid \frac{\Gamma_k(\tilde{H}_{m,n}^{-1}(y))}{q(\tilde{H}_{m,n}^{-1}(y))} \mid . \frac{q(\tilde{H}_{m,n}^{-1}(y))}{q(y)} > \frac{\epsilon}{2}\} \\ \le\ & P\{\max_{m_1 \le k \le m_2} \max_{n_1 \le l \le n_2} (\sup_{0<x\le\theta(1+e_{m,n})} \mid \frac{\Gamma_k(x)}{q(x)} \mid . \sup_{\frac{1}{k+l+1} \le y \le \theta} \frac{q(\tilde{H}_{m,n}^{-1}(y))}{q(y)}) > \frac{\epsilon}{2}\} \\ \le\ & P\{\max_{m_1 \le k \le m_2} (\sup_{0<x\le\theta(1+e_{m,n})} \mid \frac{\Gamma_k(x)}{q(x)} \mid . \max_{n_1 \le l \le n_2} \sup_{\frac{1}{k+l+1} \le y \le \theta} \frac{q(\tilde{H}_{m,n}^{-1}(y))}{q(y)}) > \frac{\epsilon}{2}\}. \end{aligned} \quad (4.58)$$

By Remark 1 of Wellner (1978) we have for some large $\alpha > 1$ and all $k+l \ge 1$, $\tilde{H}_{m,n}^{-1}(y) \le \alpha y$ uniformly in $y \in [\frac{1}{k+l+1}, 1]$ with probability arbitrarily near 1. Thus, since $q$ is assumed to be non-decreasing in a neighbourhood of zero, on taking $\theta$ small enough we get that for some large enough $\alpha > 1$ and all $k+l \ge 1$,

$$\sup_{\frac{1}{k+l+1} \le y \le \theta} \frac{q(\tilde{H}_{m,n}^{-1}(y))}{q(y)} \le \sup_{\frac{1}{k+l+1} \le y \le \theta} \frac{q(\alpha y)}{q(y)} < \infty, \quad (4.59)$$

with probability arbitrarily close to 1.

Combining (4.68) and (4.69) we get;

$$g_{2,m,n} \le P\{\max_{m_1 \le k \le m_2} \sup_{0<x\le\theta(1+o_{\text{a.s.}}(1))} \frac{\mid \Gamma_k(x) \mid}{q(x)} . \sup_{\frac{1}{m+n+1} \le y \le \theta} \frac{q(\alpha y)}{q(y)} > \frac{\epsilon}{2}\} + d_{m,n}, \quad (4.60)$$

where,

$$d_{m,n} = P\Big( \sup_{\frac{1}{k+l+1} \le y \le \theta} \frac{\hat{H}^{-1}_{m,n}(y)}{y} > \alpha \Big),$$

will be arbitrarily small for all large enough $\alpha > 1$ and small enough $\theta$.

By the definition of the process $\Upsilon_{.}(.)$ in (4.7) we also see that (4.56) becomes

$$\begin{aligned} g_{1,m,n} &= P\Big( \max_{m_1 \le k \le m_2} \max_{n_1 \le l \le n_2} \sup_{0 < y < \frac{1}{k+l+1}} \frac{\mid \Gamma_k(y) \mid}{q(y)} > \frac{\epsilon}{2} \Big) \\ &\le P\Big( \max_{m_1 \le k \le m_2} \sup_{0 < y \le \theta} \frac{\mid \Gamma_k(y) \mid}{q(y)} > \frac{\epsilon}{2} \Big). \qquad (4.61) \end{aligned}$$

From (4.55)-(4.61), we can see that (4.54) (and hence (4.50)) will be true if we show that for any $\epsilon > 0$,

$$\lim_{\theta \to 0} \limsup_{(m \wedge n) \to \infty} P\Big( \max_{m_1 \le k \le m_2} \sup_{0 < y \le \theta} \frac{\mid \Gamma_k(y) \mid}{q(y)} > \frac{\epsilon}{2} \Big) = 0, \qquad (4.62)$$

where $m_1 = m(\tau_2 - a_m)$ and $m_2 = m(\tau_2 + a_m)$ are as in (4.1).

Now, by inequality (3) of Sen in Shorack and Wellner (1986), p.139, we have

$$P\Big( \max_{1 \le k \le m} \sqrt{\frac{k}{m}} \sup_{0 < x \le \theta} \frac{\mid \Gamma_k(x) \mid}{q(x)} > \lambda \Big) \le \frac{\mu_m(\theta)}{\lambda}, \qquad \forall \lambda > 0, \qquad (4.63)$$

where, $\theta > 0$ and

$$\mu_m(\theta) = E\Big\{ \sup_{0 < x \le \theta} \frac{\mid \Gamma_m(x) \mid}{q(x)} \Big\}. \qquad (4.64)$$

Also, by inequality (1) of Pyke and Shorack in Shorack and Wellner (1986), p.134, we have for $X_m(\theta) = \sup_{0 < x \le \theta} \{ \mid \Gamma_m(x) \mid / q(x) \}$ and any $\epsilon' > 0$

$$P(X_m(\theta) > \epsilon'^2) \le \int_0^\theta (q(t))^{-2} \mathrm{d}t / \epsilon'^4 \le \epsilon'^2, \qquad (4.65)$$

by choosing $\theta$ small enough.

By (4.64), (4.65) and Exercise 2, p.43 of Chung (1974), we have for any $\epsilon' > 0$,

$$\begin{aligned} P &\ (\max_{1\le k\le m}\sqrt{\frac{k}{m}}\sup_{0<x\le\theta}\frac{\mid \Gamma_k(x)\mid}{q(x)} > \epsilon') \\ \le &\ \frac{1}{\epsilon'}\{\int_{\{X_m(\theta)>\epsilon'^2\}} X_m(\theta)dP + \int_{\{X_m(\theta)\le\epsilon'^2\}} X_m(\theta)dP\} \\ \le &\ \epsilon', \qquad\qquad \text{as } \theta\to 0 \text{ and m} \to \infty. \end{aligned} \tag{4.66}$$

Thus (4.50) is true and hence combining (4.48), (4.49) and (4.50) we finish the proof of Theorem (4.4).

## 4.4 Test statistics for the two-sample problem and their limiting distributions

Define the statistic

$$T_{m,n} = \frac{1}{M}\sum_{j=1}^{M} J_{m,n}(\frac{r_j}{M+N})$$

where $J_{m,n}(.)$, is score function and $r_j$ is the rank of the $j^{th}$ observation of the first sample among the observations of the combined sample.

Using Parzen (1983)'s representation we obtain

$$T_{m,n} = \int_0^1 J_{m,n}(y)\mathrm{d}D_{m,n}^{-1}(y), \qquad p = 1, 2, \tag{4.67}$$

where $D_{m,n}^{-1}(.)$, are defined by (4.33).

To study the asymptotic theory of the statistic $T_{m,n}$, we assume that:

(1) There exists a real function $J$ on $(0,1)$ for which $J'$ exists almost everywhere with respect to Lebesgue measure on $(0,1)$ such that,

$$\eta_{m,n} = \sqrt{m}\int_0^1 (J_{m,n}(y) - J(y))\mathrm{d}D^{-1}_{m,n}(y) \overset{\mathrm{p}}{=} \mathrm{o}(1), \text{ as } (m \wedge n) \to \infty.$$

(2) For some $q \in Q$ we have

(i) $\lim_{y\downarrow 0} q(y)J(y) = \lim_{y\uparrow 1} q(y)J(y) = O(1)$,

(ii) $\int_0^1 q(y) \mid J'(y) \mid \mathrm{d}y < \infty$.

**Theorem** (4.5)

Let $\Delta_{m,n}(.)$, be as in Theorem (4.3). Then under $H_\circ$ of (4.3), we have as $(m \wedge n) \to \infty$

$$A_{m,n} = \mid t_{m,n} - \int_0^1 J(y)\mathrm{d}\Delta_{m,n}(y) \mid \overset{\mathrm{p}}{=} \mathrm{o}(1), \qquad ,$$

where

$$t_{m,n} = \sqrt{\frac{mn}{m+n}}\{T_{m,n} - \int_0^1 J(y)\mathrm{d}y\}, \tag{4.68}$$

and $T_{m,n}$ is defined by (4.67).

**Proof**

Using condition (1) of the score function above we have ;

$$\begin{aligned} t_{m,n} &= \sqrt{\frac{mn}{m+n}}\{T_{m,n} - \int_0^1 J(y)\mathrm{d}y \pm \int_0^1 J(y)\mathrm{d}D^{-1}_{m,n}(y)\} \\ &= \sqrt{\frac{mn}{m+n}}\{\int_0^1 J(y)\mathrm{d}(D^{-1}_{m,n}(y) - y)\} + \sqrt{\frac{n}{m+n}}\eta_{m,n} \\ &= \int_0^1 J(y)\mathrm{d}R_{m,n}(y) + \mathrm{o}_{\mathrm{p}}(1), \end{aligned} \tag{4.69}$$

where $R_{m,n}(.)$ is defined by (4.34).

Let $\Delta^{*}_{m,n}(.) = -\Delta_{m,n}(.)$ be as defined in (4.37) and define the process $\{\Pi_{m,n}(y), 0 \leq y \leq 1\}$ by;

$$\Pi_{m,n}(y) = R_{m,n}(y) - \Delta_{m,n}(y), \qquad 0 \leq y \leq 1. \tag{4.70}$$

Using integration by parts, condition (2) of the score function, Theorem (4.4), (4.69) and (4.70) we obtain

$$\begin{aligned}
A_{m,n} &\leq \mid \int_0^1 J(y)\mathrm{d}R_{m,n}(y) - \int_0^1 J(y)\mathrm{d}\Delta_{m,n}(y) \mid + o_{\mathrm{p}}(1) \\
&\leq \mid \int_0^1 J(y)\mathrm{d}\Pi_{m,n}(y) \mid + o_{\mathrm{p}}(1) \\
&\leq \mid \{[J(y)\Pi_{m,n}(y)]_{y=0}^{y=1} - \int_0^1 \Pi_{m,n}(y)J'(y)\mathrm{d}y\} \mid + o_{\mathrm{p}}(1) \\
&\leq \mid [J(y)q(y).\frac{\Pi_{m,n}(y)}{q(y)}]_{y=0}^{y=1} \mid + \mid \int_0^1 \frac{\Pi_{m,n}(y)}{q(y)}.J'(y)q(y)\mathrm{d}y \mid + o_{\mathrm{p}}(1) \\
&\leq \sup_{0<y<1} \frac{\mid \Pi_{m,n}(y) \mid}{q(y)}.\{\mid \lim_{y\uparrow 1} J(y)q(y) \mid + \mid \lim_{y\downarrow 0} J(y)q(y) \mid \\
&\quad + \int_0^1 \mid J'(y) \mid q(y)\mathrm{d}y\} + o_{\mathrm{p}}(1) \\
&\overset{\mathrm{P}}{=} o(1), \qquad \text{as } (m \wedge n) \to \infty.
\end{aligned}$$

Now, consider

$$T_a = (\frac{n}{m+n})^{-1}\{t_{m,n}/\gamma(J)\}, \tag{4.71}$$

where $t_{m,n}$, is defined by (4.68) and

$$\gamma(J)^2 = 2\int_0^1 \int_0^y x(1-y)J'(x)J'(y)\mathrm{d}x\mathrm{d}y, \qquad .$$

Let $\Delta^{*}_{m,n}(.) = -\Delta_{m,n}(.)$, be as in (4.37) , then by Theorem (4.5) we have as $(m \wedge n) \to \infty$

$$| T_a - (\frac{n}{m+n})^{-1}\{\int_0^1 J(y)\mathrm{d}\Delta_{m,n}(y)/\gamma(J)\} | \xrightarrow{\mathrm{P}} 0, \qquad (4.72)$$

where $T_a$ and $\gamma(J)$, are defined by (4.71).

To simplify the limiting distribution in (4.72) we argue as follows. Let $\mathcal{K}_1(.,.)$, and $\mathcal{K}_2(.,.)$, be the independent Kiefer processes of Theorem (4.1) and recall that

$$\Delta_{m,n}(y) = -(\frac{n\tau_1}{m\tau_2 + n\tau_1})\{(\frac{m}{m+n})^{\frac{1}{2}}\frac{\mathcal{K}_1(y, n\tau_1)}{\tau_1\sqrt{n}} - (\frac{n}{m+n})^{\frac{1}{2}}\frac{\mathcal{K}_2(y, m\tau_2)}{\tau_2\sqrt{m}}\}. \qquad (4.73)$$

It is easy to see that;

$$Cov\{\Delta_{m,n}(y_1), \Delta_{m,n}(y_2)\} = (\frac{n}{m+n})(\frac{n\tau_1}{m\tau_2 + n\tau_1})(\frac{1}{\tau_2})(y_1 \wedge y_2) - y_1y_2)$$

Hence for all $m, n \geq 1$ we have

$$\Delta_{m,n}(y) \overset{\mathrm{d}}{=} \{(\frac{n}{m+n})(\frac{n\tau_1}{m\tau_2 + n\tau_1})(\frac{1}{\tau_2})\}^{1/2}B(y), \qquad (4.74)$$

where $B(.)$ is a standard Brownian bridge.

Combining (4.72)and (4.74) we have

$$| T_a - \{(\frac{n\tau_1}{m\tau_2 + n\tau_1})(\frac{1}{\tau_2})\}^{1/2}\int_0^1 J(y)\mathrm{d}B(y)/\gamma(J) | \xrightarrow{\mathrm{P}} 0. \qquad (4.75)$$

Hence,

$$\hat{T}_{m,n} := \{(1 + \frac{M}{N})(\frac{M}{m})\}^{1/2}T_a \xrightarrow{\mathrm{d}} \int_0^1 J(y)\mathrm{d}B(y)/\gamma(J), \qquad (4.76)$$

where by Remark 5.2 of Aly *et al.* (1987), the limiting variable of (4.76) is $\mathcal{N}(0,1)$.

### 4.5 Change point tests when the sample size is random

The main contribution of this section is the development of tests for a change point when the sample size is random. We start by introducing the test processes, representing them in terms of empirical and quantile processes and then derive their limiting distributions.

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with continuous distribution function $F(.)$ and $\{N_n, n \geq 1\}$ be a sequence of nonnegative integer-valued random variables. Now, suppose that $X_1, X_2, \ldots, X_{N_n}$ is a sample of a random size $N_n, n \geq 1$. We study here the problem of testing the null hypothesis of no change against the at most one change point alternative,

$$H_1 \ : \ X_i \sim F \ , \ 1 \leq i \leq k \ \& \ X_i \sim G \ , i > k, \tag{4.77}$$

where $F \neq G$ and $k$ are unknown.

Let $J^*(.)$ be an arbitrary score function. For $0 \leq t \leq 1$, define

$$T_n^*(t) = \frac{\sqrt{n}}{\sigma(J)}\{\frac{1}{N_n}\sum_{i=1}^{[N_n t]} J^*(\frac{r_i}{N_n}) - \frac{[N_n t]}{N_n}\mu\}, \tag{4.78}$$

where $r_i$ is the rank of the $i^{th}$ observation in the sample, $\mu = \int_0^1 J^*(y)\mathrm{d}y$, and

$$\sigma^2(J) = 2\int_0^1 \int_0^y x(1-y)J^{*\prime}(x)J^{*\prime}(y)\mathrm{d}x\mathrm{d}y. \tag{4.79}$$

As in the two-sample problem we will assume that the random variables $N_n$, $n \geq 1$ have the following property. There exist sequences $a_n$ and $b_n$ of real numbers such that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ and

$$P(\mid \frac{N_n}{n} - \tau \mid > a_n) \leq b_n, \tag{4.80}$$

where $0 < \tau \leq 1$ is a constant.

From now on we will always write $N$ instead of $N_n$ to simplify the equations. Let $\hat{F}_m(.)$ be the empirical distribution function based on the first $m$ of the observations $F(X_1), F(X_2), \ldots$ and $\hat{F}_m^{-1}(.)$ be the corresponding empirical quantile function. Define the process

$$R_n^*(y,t) \;=\; n^{\frac{1}{2}}\{\hat{F}_{[Nt]}\hat{F}_N^{-1}(y) - y\}, \qquad 0 \leq y \leq 1, 0 \leq t \leq 1. \qquad (4.81)$$

Using (4.81) and Parzen (1983) we represent (4.78) as;

$$\sigma_{(J)}T_n^*(t) = \frac{[Nt]}{N}\int_0^1 J^*(y)\mathrm{d}R_n^*(y,t), \qquad 0 \leq t \leq 1. \qquad (4.82)$$

By the definition of the process $R_n^*(.,.)$ in (4.81), we have for $0 \leq y \leq 1$ and $0 \leq t \leq 1$;

$$\begin{aligned}\frac{[Nt]}{N}R_n^*(y,t) &= \frac{[Nt]}{N}\sqrt{n}\{\hat{F}_{[Nt]}\hat{F}_N^{-1}(y) - y\}\\ &= \frac{\sqrt{n}}{N}[Nt]^{\frac{1}{2}}\{\Gamma_{[Nt]}(\hat{F}_N^{-1}(y)) - (\frac{[Nt]}{N})^{\frac{1}{2}}\Gamma_N(y)\\ &\quad -(\frac{[Nt]}{N})^{\frac{1}{2}}(\Upsilon_N(y) - \Gamma_N(y))\}, \qquad (4.83)\end{aligned}$$

where $\Gamma_{.}(.)$ and $\Upsilon_{.}(.)$ are defined by (4.6) and (4.7) respectively.

Next, we shall show that the third term of the R.H.S. of (4.83) is of a small order in probability.

**Lemma** (4.3)

Let $\Gamma_{.}(.)$ and $\Upsilon_{.}(.)$ be defined by (4.6) and (4.7), respectively. Then, under $H_o$ we have as $n \to \infty$

$$\sup_{0\leq s\leq 1}\sup_{0\leq y\leq 1} \mid (\frac{n}{N})^{\frac{1}{2}}(\frac{[Ns]}{N})(\Upsilon_N(y) - \Gamma_N(y)) \mid \overset{P}{=} o(1).$$

**Proof**

By (4.80) we get for any $\epsilon > 0$

$$\begin{aligned}
& P\{\sup_{0\le s\le 1}(\frac{n}{N})^{\frac{1}{2}}(\frac{[Ns]}{N})\sup_{0\le y\le 1}\mid \Upsilon_N(y)-\Gamma_N(y))\mid > \epsilon\} \\
\le\ & P\{\sup_{0\le s\le 1}(\frac{[Ns]}{n})(\frac{n}{N})^{\frac{3}{2}}\sup_{0\le y\le 1}\mid \Upsilon_N(y)-\Gamma_N(y))\mid > \epsilon, \mid \frac{N}{n}-\tau \mid \le a_n\}+b_n \\
\le\ & P\{(\tau-a_n)^{-\frac{3}{2}}\max_{n_1\le m\le n_2}\sup_{0\le y\le 1}\mid \Upsilon_m(y)-\Gamma_m(y))\mid > \epsilon\}+b_n, \qquad (4.84)
\end{aligned}$$

where $n_1 = n(\tau - a_n)$ and $n_2 = n(\tau + a_n)$.

Using Theorem (3.3.1) of Csörgő and Horváth (1993), we have as $n \to \infty$

$$\begin{aligned}
\max_{n_1\le m\le n_2}\sup_{0\le y\le 1}\mid \Upsilon_m(y)-\Gamma_m(y))\mid &\le \max_{m\ge n_1}\sup_{0\le y\le 1}\mid \Upsilon_m(y)-\Gamma_m(y))\mid \\
&\le \overline{\lim_{m\to\infty}}\sup_{0\le y\le 1}\mid \Upsilon_m(y)-\Gamma_m(y))\mid \\
&\overset{\text{a.s.}}{=} o(1). \qquad (4.85)
\end{aligned}$$

Combining (4.84) and (4.85) we finish the proof of the Lemma.

Next we approximate the first two terms of (4.84), by Gaussian processes.

**Lemma** (4.4)

Under $H_\circ$, there exists a Kiefer process $\mathcal{K}(.,.)$ such that , as $n \to \infty$

$$d_{1n} = \sup_{0\le s\le 1}\sup_{0\le y\le 1}\mid (\frac{[Ns]}{N})(\frac{n}{N})^{\frac{1}{2}}\Gamma_N(y) - s\mathcal{K}(y,n\tau)/(\tau\sqrt{n})\mid \overset{\text{P}}{=} o(1).$$

**Proof**

We can see that

$$\begin{aligned}
d_{1n} &\le \sup_{0\le s\le 1}(\frac{[Ns]}{N})\sup_{0\le y\le 1}\mid (\frac{n}{N})N^{\frac{1}{2}}\Gamma_N(y)-\frac{1}{\tau}\mathcal{K}(y,n\tau)\mid /\sqrt{n} \\
&\quad + \sup_{0\le s\le 1}\mid \frac{[Ns]}{N}-s\mid . \sup_{0\le y\le 1}\mid \mathcal{K}(y,n\tau)\mid /(\tau\sqrt{n}) \\
&\le d_{1n}^{(1)}+d_{1n}^{(2)}+d_{1n}^{(3)}, \qquad (4.86)
\end{aligned}$$

where,

$$d_{1n}^{(1)} = \sup_{0\le s\le 1} (\frac{[Ns]}{N}) \sup_{0\le y\le 1} \mid N^{\frac{1}{2}}\Gamma_N(y) - \mathcal{K}(y, n\tau) \mid (\frac{n}{N})/\sqrt{n}, \tag{4.87}$$

$$d_{1n}^{(2)} = \sup_{0\le s\le 1} (\frac{[Ns]}{N}) \sup_{0\le y\le 1} \mid \mathcal{K}(y, n\tau) \mid /\sqrt{n}.\{\mid \frac{n}{N} - \frac{1}{\tau} \mid\}, \tag{4.88}$$

and

$$d_{1n}^{(3)} = \sup_{0\le s\le 1} \mid \frac{[Ns]}{N} - s \mid . \sup_{0\le y\le 1} \mid \mathcal{K}(y, n\tau) \mid /(\tau\sqrt{n}). \tag{4.89}$$

Form (4.17), (4.18) and (4.19) we can see that

$$d_{1n}^{(1)} \overset{\text{p}}{=} \mathrm{o}(1), \qquad \text{as } n \to \infty \tag{4.90}$$

and

$$d_{1n}^{(2)} \overset{\text{p}}{=} \mathrm{o}(1), \qquad \text{as } n \to \infty. \tag{4.91}$$

By the scale transformation of the Kiefer process we have for every $n \ge 1$;

$$\sup_{0\le y\le 1} \mid \mathcal{K}(y, n\tau) \mid /\sqrt{n\tau} \overset{\text{d}}{=} \sup_{0\le y\le 1} \mid B_1^{\circ}(y) \mid \overset{\text{p}}{=} O(1), \tag{4.92}$$

where $B_1^{\circ}(.)$ is a standard Brownian bridge.

Using (4.80) and (4.92) we have for any $\epsilon > 0$

$$\begin{aligned} P\{d_{1n}^{(3)} > \epsilon\} &\le P\{d_{1n}^{(3)} > \epsilon, \mid \frac{N}{n} - \tau \mid < a_n\} + b_n \\ &\le P\{\sup_{0\le s\le 1} \mid [Ns] - Ns \mid \frac{(\tau\sqrt{n})^{-1}}{n(\tau - a_n)} \sup_{0\le y\le 1} \mid \mathcal{K}(y, n\tau) \mid > \epsilon\} + b_n \\ &\le P\{\frac{n^{-1}}{(\tau - a_n)} \sup_{0\le y\le 1} \mid \mathcal{K}(y, n\tau) \mid /(\tau\sqrt{n}) > \epsilon\} + b_n \\ &= \mathrm{o}(1), \qquad \text{as}\, n \to \infty. \end{aligned} \tag{4.93}$$

Combining (4.86), (4.87), (4.88) and (4.93) we get the proof of the Lemma.

**Lemma** (4.5)

Let $\Gamma_{.}(.)$ be defined by (4.6), then under $H_\circ$ we have as $n \to \infty$

$$d_{2n} = \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid (\frac{[Ns]}{N})^{\frac{1}{2}} (\frac{n}{N})^{\frac{1}{2}} \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) - \frac{n^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, ns\tau) \mid \overset{\mathrm{P}}{=} \circ(1),$$

where $\mathcal{K}(.,.)$ is the Kiefer process of Lemma (4.4).

**Proof**

First we notice that

$$\begin{aligned}
d_{2n} \quad \le & \quad \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid (\frac{n}{N})[Ns]^{\frac{1}{2}} \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) - \frac{1}{\tau} \mathcal{K}(y, [ns\tau]) \mid / \sqrt{n} \\
& + \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid \mathcal{K}(y, ns\tau) - \mathcal{K}(y, [ns\tau]) \mid / (\tau \sqrt{n}) \\
\le & \quad \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid (\frac{n}{N})[Ns]^{\frac{1}{2}} \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) - \frac{1}{\tau} \mathcal{K}(y, [Ns]) \mid / \sqrt{n} \\
& + \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid \mathcal{K}(y, [ns\tau]) - \mathcal{K}(y, [Ns]) \mid / (\tau \sqrt{n}) \\
& + \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid \mathcal{K}(y, [ns\tau]) - \mathcal{K}(y, ns\tau) \mid / (\tau \sqrt{n}) \\
\le & \quad d_{2n}^{(1)} + d_{2n}^{(2)} + d_{2n}^{(3)} + d_{2n}^{(4)} + d_{2n}^{(5)},
\end{aligned} \tag{4.94}$$

where,

$$d_{2n}^{(1)} = \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} (\frac{N}{n})^{-1} \mid [Ns]^{\frac{1}{2}} \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) - \mathcal{K}(\hat{F}_N^{-1}(y), [Ns]) \mid n^{-\frac{1}{2}}, \tag{4.95}$$

$$d_{2n}^{(2)} = \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} (\frac{N}{n})^{-1} \mid \mathcal{K}(y, [Ns]) - \mathcal{K}(\hat{F}_N^{-1}(y), [Ns]) \mid / \sqrt{n}, \tag{4.96}$$

$$d_{2n}^{(3)} = \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} (\frac{N}{n})^{-1} \mid \mathcal{K}(y, [ns\tau]) - \mathcal{K}(y, [Ns]) \mid / \sqrt{n} \tag{4.97}$$

$$d_{2n}^{(4)} = \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid \mathcal{K}(y, [ns\tau]) - \mathcal{K}(y, ns\tau) \mid / (\tau \sqrt{n}) \tag{4.98}$$

and

$$d_{2n}^{(5)} = \sup_{0\le s\le 1}\sup_{0\le y\le 1} \mid \mathcal{K}(y, ns\tau) \mid /\sqrt{n}.\{\mid \frac{(N/n)-\tau}{\tau\,(N/n)} \mid\}. \tag{4.99}$$

Let $\epsilon > 0$, then by (4.80) we have

$$\begin{aligned}
P(d_{2n}^{(5)} > \epsilon) &\le P(d_{2n}^{(5)} > \epsilon, \mid \tau - \frac{N}{n} \mid \le a_n) + b_n \\
&\le P\{n^{-\frac{1}{2}} \sup_{0\le s\le 1}\sup_{0\le y\le 1} \mid \mathcal{K}(y, ns\tau) \mid .(\mid \frac{a_n}{\tau\,(\tau - a_n)} \mid) > \epsilon\} + b_n \\
&\le P\{\sup_{0\le s\le 1}\sup_{0\le y\le 1} \mid \mathcal{K}^{\circ}(y, \frac{ns\tau}{n}) \mid .(\mid \frac{a_n}{\tau\,(\tau - a_n)} \mid) > \epsilon\} + b_n \\
&\le P\{\sup_{0\le x\le 1}\sup_{0\le y\le 1} \mid \mathcal{K}^{\circ}(y, x) \mid .(\mid \frac{a_n}{\tau\,(\tau - a_n)} \mid) > \epsilon\} + b_n \\
&= o(1), \qquad\qquad \text{as } n \to \infty,
\end{aligned} \tag{4.100}$$

where $\mathcal{K}^{\circ}(.,.)$ is a standard Kiefer process.

By Theorem A and (4.80) we get for any $\epsilon > 0$

$$\begin{aligned}
P\{d_{2n}^{(1)} > \epsilon\} &\le P\{d_{2n}^{(1)} > \epsilon, \mid \frac{N}{n} - \tau \mid \le a_n\} + b_n \\
&\le P\{\frac{n^{-\frac{1}{2}}}{(\tau - a_n)} \sup_{0\le s\le 1}\sup_{0\le x\le 1} \mid [Ns]^{\frac{1}{2}}\Gamma_{[Ns]}(x) - \mathcal{K}(x, [Ns]) \mid > \epsilon\} + b_n \\
&\le P\{\frac{n^{-\frac{1}{2}}}{(\tau - a_n)} \max_{1\le m\le n}\sup_{0\le x\le 1} \mid m^{\frac{1}{2}}\Gamma_m(x) - \mathcal{K}(x, m) \mid > \epsilon\} + b_n \\
&= o(1), \qquad\qquad \text{as n} \to \infty.
\end{aligned} \tag{4.101}$$

Now, since $\tau > 0$, then from (4.80), we see that, $n_1 = n(\tau - a_n), n_2 = n(\tau + a_n) \to \infty$ as $n \to \infty$. Hence by the LIL of the quantile process (see Shorack and Wellner (1986)), we have as $n \to \infty$

$$\max_{n_1\le l\le n_2}\sup_{0\le y\le 1} \mid y - \hat{F}_l^{-1}(y) \mid$$

$$
\begin{aligned}
&\leq \max_{n_1 \leq l \leq n_2} \left(\frac{\log\log l}{l}\right)^{\frac{1}{2}} . \max_{n_1 \leq l \leq n_2} \sup_{0 \leq y \leq 1} \mid y - \hat{F}_l^{-1}(y) \mid \left(\frac{l}{\log\log l}\right)^{\frac{1}{2}} \\
&\leq \max_{n_1 \leq l \leq n_2} \left(\frac{\log\log l}{l}\right)^{\frac{1}{2}} . \max_{l \geq n_1} \sup_{0 \leq y \leq 1} \frac{\mid \Upsilon_l(y) \mid}{\sqrt{\log\log l}} \\
&\leq \max_{l \geq n_1} \left(\frac{\log\log l}{l}\right)^{\frac{1}{2}} \overline{\lim_{l \to \infty}} \sup_{0 \leq y \leq 1} \frac{\mid \Upsilon_l(y) \mid}{\sqrt{\log\log l}} \\
&= O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}) \qquad \text{a.s..} \qquad (4.102)
\end{aligned}
$$

Let $h_n = C(\frac{\log\log n)}{n})^{\frac{1}{2}}$, $C > 0$, then using Lemma (1.11.3) of Csörgő and Révész (1981), (4.80) and (4.102) we obtain for any $\epsilon > 0$

$$
\begin{aligned}
P_{2n} &= P\{d_{2n}^{(2)} > \epsilon\} \\
&\leq P\{d_{2n}^{(2)} > \epsilon, \mid \frac{N}{n} - \tau \mid \leq a_n\} + b_n \\
&\leq P\{\frac{n^{-\frac{1}{2}}}{(\tau - a_n)} \sup_{0 \leq s \leq 1} \sup_{0 \leq y \leq 1} \mid \mathcal{K}(y, [Ns]) - \mathcal{K}(\hat{F}_N^{-1}(y), [Ns]) \mid > \epsilon\} + b_n \\
&\leq P\{\frac{n^{-\frac{1}{2}}}{(\tau - a_n)} \max_{1 \leq m \leq n} \sup_{0 \leq x \leq h_n} \sup_{0 \leq y < 1} \mid \mathcal{K}(y, m) - \mathcal{K}(y + x, m) \mid > \epsilon\} + 2b_n \\
&\leq P\{(\tau - a_n)^{-1} \max_{0 \leq \frac{m}{n} \leq 1} \sup_{0 \leq x \leq h_n} \sup_{0 \leq y < 1} \mid \mathcal{K}^*(y, \frac{m}{n}) - \mathcal{K}^*(y + x, \frac{m}{n}) \mid > \epsilon\} + 2b_n \\
&= o(1), \qquad \text{as } n \to \infty, \qquad (4.103)
\end{aligned}
$$

where $\mathcal{K}^*(.,.)$ is a standard Kiefer process.

For (4.97), we first note that if $\mid \frac{N}{n} - \tau \mid \leq a_n$ then for all $0 \leq s \leq 1$,

$$
\begin{aligned}
\mid [ns\tau] - [Ns] \mid &\leq \mid [ns\tau] - ns\tau \mid + \mid ns\tau - Ns \mid + \mid Ns - [Ns] \mid \\
&\leq 1 + na_n + 1 := \delta_n^*. \qquad (4.104)
\end{aligned}
$$

Let $\mathcal{K}^{**}(.,.)$ be a standard Kiefer process. Using (4.80), (4.104) and Mirzakhmedov and Tursunov (1992), we get for any $\epsilon > 0$

$$
P\{d_{2n}^{(3)} \; > \; \epsilon\}
$$

$$\begin{aligned}
&\leq \; P\{d_{2n}^{(3)} > \epsilon, \mid \frac{N}{n} - \tau \mid \leq a_n\} + b_n \\
&\leq \; P\{\frac{n^{-\frac{1}{2}}}{(\tau - a_n)} \sup_{0 \leq x \leq \delta_n^*} \max_{1 \leq k \leq n} \sup_{0 \leq y \leq 1} \mid \mathcal{K}(y,k) - \mathcal{K}(y,k+x) \mid > \epsilon\} + b_n \\
&\leq \; P\{\frac{1}{(\tau - a_n)} \sup_{0 \leq x \leq \frac{\delta_n^*}{n}} \sup_{0 \leq z < 1} \sup_{0 \leq y \leq 1} |\mathcal{K}^{**}(y,z) - \mathcal{K}^{**}(y,z+x)| > \epsilon\} + b_n \\
&= \; \mathrm{o}(1), \qquad \text{as } n \to \infty. \qquad (4.105)
\end{aligned}$$

Similarly we can show that;

$$d_{2n}^{(4)} \overset{\mathrm{p}}{=} \mathrm{o}(1), \qquad \text{as } n \to \infty. \qquad (4.106)$$

Substituting from (4.100), (4.101), (4.103), (4.105) and (4.106) in (4.94) we finish the proof of the Lemma.

**Theorem** (4.6)

Under $H_\circ$ of (4.77), there exists a Kiefer process $\mathcal{K}(.,.)$ such that as,$n \to \infty$

$$\sup_{0 \leq s \leq 1} \sup_{0 \leq y \leq 1} \mid (\frac{[Ns]}{N}) R_n^*(y,s) - \frac{n^{-\frac{1}{2}}}{\tau}\{\mathcal{K}(y,ns\tau) - s\mathcal{K}(y,n\tau)\} \mid \overset{\mathrm{p}}{=} \mathrm{o}(1),$$

where $R_n^*(.,.)$ is defined by (4.81). The proof of this theorem is an immediate result of Lemmas (4.3), (4.4) and (4.5).

Now, assume that $Q^*$ is the class of positive functions $q$ on $(0,1)$, that are non-decreasing in a neighbourhood of 0 and non-increasing in a neighbourhood of 1 and $\frac{\{t(1-t)\}^\nu}{q(t)} < \infty, \forall t \in (0,1), \nu < \frac{1}{2}$. We also assume that for

$$(i) \lim_{y \downarrow 0} q(y) J^*(y) = \lim_{y \uparrow 1} q(y) J^*(y) = 0, \qquad (4.107)$$

$$(ii) \int_0^1 q(y) \mid J^{*'}(y) \mid \mathrm{d}y < \infty, \qquad (4.108)$$

$$(iii) \int_0^\theta q(y) \mid J^{*'}(y) \mid \mathrm{d}y = \int_{1-\theta}^1 q(y) \mid J^{*'}(y) \mid \mathrm{d}y = 0, \;\; \text{as } \theta \searrow 0. \qquad (4.109)$$

where $q \in Q^*$, and $J^*(.)$, is the score function of (4.82).

**Lemma** (4.6)

Let $R_n^*(.,.)$ be defined by (4.81), then as $n \to \infty$

$$\epsilon_{n1} = \sup_{0 \le s \le 1} (\frac{[Ns]}{N}) \mid \int_{[0,\frac{1}{n+1})} J^*(y) \mathrm{d} R_n^*(y,s) \mid \overset{P}{=} o(1)$$

and

$$\epsilon_{n1} = \sup_{0 \le s \le 1} (\frac{[Ns]}{N}) \mid \int_{(1-\frac{1}{n+1},1]} J^*(y) \mathrm{d} R_n^*(y,s) \mid \overset{P}{=} o(1).$$

**Proof**

Here we prove only the result of $\epsilon_{n1}$, since the result for $\epsilon_{n2}$ can be done similarly.

By the representation of the process $R_n^*(.,.)$ in (4.83) we have

$$\epsilon_{n1} \le \epsilon_{n1}^{(1)} + \epsilon_{n1}^{(2)}, \tag{4.110}$$

where,

$$\epsilon_{n1}^{(1)} = \sup_{0 \le s \le 1} \mid \int_{[0,\frac{1}{n+1})} J^*(y) \mathrm{d}\{\frac{(n[Ns])^{\frac{1}{2}}}{N} \Gamma_{[Ns]}(\hat{F}_N^{-1}(y))\} \mid \tag{4.111}$$

and

$$\epsilon_{n1}^{(2)} = \sup_{0 \le s \le 1} \mid \int_{[0,\frac{1}{n+1})} J^*(y) \mathrm{d}\{(\frac{n^{\frac{1}{2}}[Ns]}{N^{\frac{3}{2}}}) \Upsilon_N(y)\} \mid . \tag{4.112}$$

Using integration by parts we have by (4.7), (4.107) and (4.108);

$$\epsilon_{n1}^{(1)} = \sup_{0 \le s \le 1} (\frac{(n[Ns])^{\frac{1}{2}}}{N}) \{ \mid [J^*(y) q(y). \frac{\Gamma_{[Ns]}(\hat{F}_N^{-1}(y))}{q(y)}]_{y=0}^{y=\frac{1}{n+1}}$$

$$\begin{aligned} & - \int_{[0,\frac{1}{n+1})} \frac{\Gamma_{[Ns]}(\hat{F}_N^{-1}(y))}{q(y)} . J^{*\prime}(y) q(y) \mathrm{d}y \mid \} \\ \leq \ & C_1 . \sup_{0 \leq s \leq 1} \left( \frac{(n[Ns])^{\frac{1}{2}}}{N} \right) . \sup_{0<y<\frac{1}{n+1}} \frac{\mid \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) \mid}{q(y)} \\ \leq \ & C_1 . \sup_{0 \leq s \leq 1} \sup_{0<y<\frac{1}{n+1}} \left( \frac{N}{n} \right)^{-1} . \left( \frac{[Ns]}{n} \right)^{\frac{1}{2}} \frac{\mid \Gamma_{[Ns]}(\hat{F}_N^{-1}(y)) \mid}{q(y)} \\ \leq \ & C_1 . \max_{1 \leq l \leq n} \sup_{0<y<\frac{1}{n+1}} \left( \frac{N}{n} \right)^{-1} . \left( \frac{l}{n} \right)^{\frac{1}{2}} \frac{\mid \Gamma_l(\hat{F}_N^{-1}(y)) \mid}{q(y)}, \qquad (4.113) \end{aligned}$$

where, $\{2 \mid \lim_{y \downarrow 0} J^*(y) q(y) \mid + \int_0^1 \mid J^{*\prime}(y) \mid q(y) \mathrm{d}y\} \leq C_1 < \infty$.

Let $\theta = \frac{1}{n+1}$ in (4.63)-(4.66), then by (4.80) we get for any $\epsilon > 0$,

$$\begin{aligned} P(e_{n1}^{(1)} > \epsilon) \ & \leq \ P(e_{n1}^{(1)} > \epsilon, \mid \frac{N}{n} - \tau \mid \leq a_n) + b_n \\ & \leq \ P\{C_1 . \max_{1 \leq l \leq n} \sup_{0<y<\frac{1}{n+1}} \left( \frac{l}{n} \right)^{\frac{1}{2}} \frac{\mid \Gamma_l(y) \mid}{q(y)} . (\tau - a_n)^{-1} > \epsilon\} + b_n \\ & = \ o(1), \qquad \text{as } n \to \infty. \qquad (4.114) \end{aligned}$$

For $e_{n1}^{(2)}$ of (4.110) we have by the definition of $\Upsilon_{.}(.)$ in (4.7),

$$P(e_{n1}^{(2)} > \epsilon) = 0, \qquad \forall \epsilon > 0. \qquad (4.115)$$

By (4.110), (4.114) and (4.115) we finish the required proof.

**Lemma** (4.7)

Let $\mathcal{K}(.,.)$ be a Kiefer process. Then as $\theta \searrow 0$

$$\eta_{n1} = \sup_{0<s<1} \mid \int_{[0,\theta)} J^*(y) \mathrm{d}\{ \frac{n^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, ns\tau) - \frac{sn^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, n\tau)\} \mid \overset{\mathrm{P}}{=} o(1)$$

and

$$\eta_{n2} = \sup_{0<s<1} \mid \int_{(1-\theta,1]} J^*(y) \mathrm{d}\{ \frac{n^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, ns\tau) - \frac{sn^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, n\tau)\} \mid \overset{\mathrm{P}}{=} o(1),$$

where $J^*(.)$ satisfies (4.107)-(4.109).

**Proof**

As in Lemma (4.6), we prove here only the result of $\eta_{n1}$. We note first that

$$\eta_{n1} \leq \eta_{n1}^{(1)} + \eta_{n1}^{(2)}, \tag{4.116}$$

where,

$$\eta_{n1}^{(1)} = \sup_{0<s<1} \mid \int_{[0,\theta)} J^*(y) \mathrm{d}\{\frac{n^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, ns\tau)\} \mid$$

and

$$\eta_{n1}^{(2)} = \sup_{0<s<1} \mid \int_{[0,\theta)} J^*(y) \mathrm{d}\{\frac{sn^{-\frac{1}{2}}}{\tau} \mathcal{K}(y, n\tau)\} \mid .$$

Using integration by parts and (4.107)-(4.109) we get as $\theta \searrow 0$

$$\begin{aligned}
\eta_{n1}^{(1)} &= \sup_{0<s<1} \mid [J^*(y)q(y). \frac{\mathcal{K}(y, ns\tau)}{\tau\sqrt{n}q(y)}]_{y=0}^{y=\theta} - \int_{[0,\theta)} \frac{\mathcal{K}(y, ns\tau)}{\tau\sqrt{n}q(y)}. J^{*\prime}(y)q(y)\mathrm{d}y \mid \\
&\leq \sup_{0<s<1} \sup_{0<y<1} \frac{\mid \mathcal{K}(y, ns\tau) \mid}{\tau\sqrt{n}q(y)}. \{2 \mid \lim_{y\downarrow 0} J^*(y)q(y) \mid + \int_{[0,\theta)} \mid J^{*\prime}(y) \mid q(y)\mathrm{d}y\} \\
&\leq \frac{1}{\sqrt{\tau}} \sup_{0<s<1} \sup_{0<y<1} \frac{\mid \mathcal{K}^\circ(y, s) \mid}{q(y)}. \{2 \mid \lim_{y\downarrow 0} J^*(y)q(y) \mid + \int_{[0,\theta)} \mid J^{*\prime}(y) \mid q(y)\mathrm{d}y\} \\
&= O_{\mathrm{p}}(1)\{0 + \mathrm{o}(1)\} \overset{\mathrm{p}}{=} \mathrm{o}(1), \qquad (4.117)
\end{aligned}$$

where $\mathcal{K}^\circ(.,.)$ is a Kiefer process. Similarly, as $\theta \searrow 0$ we get

$$\eta_{n1}^{(2)} \overset{\mathrm{p}}{=} \mathrm{o}(1). \tag{4.118}$$

Substituting (4.117) and (4.118) in (4.116) we get

$$\eta_{n1} \overset{\mathrm{p}}{=} \mathrm{o}(1) \qquad \text{as } \theta \searrow 0,$$

which complete the required proof.

Next we introduce the main Theorem of this section in which we approximate the process $T_n^*(.)$ of (4.78) by Gaussian processes.

**Theorem** (4.7)

Let $T_n^*(.)$ be defined by (4.78). Then there exists a Kiefer process $\mathcal{K}(.,.)$ such that

$$\Pi_{n1} = \sup_{0<s<1} \mid T_n^*(s) - \int_0^1 J^*(y) \mathrm{d}\{\frac{1}{\tau\sigma(J)}(\mathcal{K}(y, ns\tau) - s\mathcal{K}(y, n\tau))\}/\sqrt{n} \mid \overset{P}{=} o(1)$$

where $\sigma(J)$ is defined by (4.78).

**Proof**

Note that

$$\Pi_{n1} = \sup_{0<s<1} \mid \int_0^1 J^*(y) \mathrm{d}\hat{\Delta}_n(y, s) \mid, \tag{4.119}$$

where,

$$\hat{\Delta}_n(y, s) = \frac{1}{\sigma(J)}(\frac{[Ns]}{N}).R_n^*(y, s) - \frac{n^{-\frac{1}{2}}}{\sigma(J)\tau}\{\mathcal{K}(y, ns\tau) - s\mathcal{K}(y, n\tau)\}. \tag{4.120}$$

It is easy to see that

$$\Pi_{n1} \leq \Pi_{n1}^{(1)} + \Pi_{n1}^{(2)} + \Pi_{n1}^{(3)} + \Pi_{n1}^{(4)} + \Pi_{n1}^{(5)}, \tag{4.121}$$

where,

$$\Pi_{n1}^{(i)} = \sup_{0<s<1} \mid \int_{y \in I_{ni}} J^*(y) \mathrm{d}\hat{\Delta}_n(y, s) \mid, \qquad i = 1, 2, \ldots, 5, \tag{4.122}$$

and $I_{ni}$, $i = 1, 2, \ldots, 5$ are the intervals $I_{n1} = [0, \frac{1}{n+1})$, $I_{n2} = [\frac{1}{n+1}, \theta)$, $I_{n3} = [\theta, 1-\theta]$, $I_{n4} = (1-\theta, 1-\frac{1}{n+1}]$ and $I_{n5} = (1-\frac{1}{n+1}, 1]$.

Using integration by parts, Theorem (4.6), (4.107)-(4.109) and the fact that $\inf_{\theta \leq y \leq 1-\theta} q(y) > 0$, for any $\theta \in (0, \frac{1}{2}]$, we get as $n \to \infty$

$$\begin{aligned}
\Pi_{n1}^{(3)} &= \sup_{0<s<1} \mid [J^{*}(y)q(y).\frac{\hat{\Delta}_n(y,s)}{q(y)}]_{y=\theta}^{y=1-\theta} - \int_{y \in I_{n3}} \frac{\hat{\Delta}_n(y,s)}{q(y)}.J^{*\prime}(y)q(y)\mathrm{d}y \mid \\
&\leq \sup_{0<s<1} \sup_{\theta \leq y \leq 1-\theta} \frac{\mid \hat{\Delta}_n(y,s) \mid}{q(y)}.\{\mid J^{*}(\theta)q(\theta) \mid + \mid J^{*}(1-\theta)q(1-\theta) \mid \\
&\quad + \int_0^1 \mid J^{*\prime}(y) \mid q(y)\mathrm{d}y\} \\
&\leq \sup_{0<s<1} \sup_{0<y<1} \mid \hat{\Delta}_n(y,s) \mid . \sup_{\theta \leq y \leq 1-\theta} q^{-1}(y).C_2(\theta) \overset{\mathrm{P}}{=} \mathrm{o}(1), \qquad (4.123)
\end{aligned}$$

where $C_2(\theta) < \infty$, for any $\theta \in (0, \frac{1}{2}]$.

From Lemmas (4.6) and (4.7) we obtain, as $n \to \infty$

$$\Pi_{n1}^{(1)} \leq e_{n1} + \eta_{n1} \overset{\mathrm{P}}{=} \mathrm{o}(1) \qquad (4.124)$$

and

$$\Pi_{n1}^{(5)} \leq e_{n2} + \eta_{n2} \overset{\mathrm{P}}{=} \mathrm{o}(1). \qquad (4.125)$$

For $\Pi_{n1}^{(2)}$ of (4.121) we have by Lemma (4.7)

$$\begin{aligned}
\Pi_{n1}^{(2)} &\leq \sup_{0<s<1} \mid \int_{y \in I_{n2}} J^{*}(y)\mathrm{d}\{(\frac{[Ns]}{N}).R_n^{*}(y,s)\} \mid + \eta_{n1} \\
&\leq \sup_{0<s<1} \mid \int_{y \in I_{n2}} J^{*}(y)\mathrm{d}\{(\frac{[Ns]}{N}).R_n^{*}(y,s)\} \mid + \mathrm{o_p}(1). \qquad (4.126)
\end{aligned}$$

We also note that by (4.83)

$$\sup_{0<s<1} \mid \int_{y \in I_{n2}} J^{*}(y)\mathrm{d}\{(\frac{[Ns]}{N}).R_n^{*}(y,s)\} \mid \leq \nu_{n1} + \nu_{n2}, \qquad (4.127)$$

where,

$$\nu_{n1} = \sup_{0 \leq s \leq 1} \mid \int_{y \in I_{n2}} J^{*}(y)\mathrm{d}\{(\frac{[Ns]}{N}).\Gamma_{[Ns]}(\hat{F}_N^{-1}(y))\} \mid \qquad (4.128)$$

and

$$\nu_{n2} = \sup_{0\leq s\leq 1} \mid \int_{y\in I_{n2}} J^*(y)\mathrm{d}\{(\frac{\sqrt{n}[Ns]}{N^{\frac{3}{2}}})\Upsilon_N(y)\} \mid . \tag{4.129}$$

Using integration by parts and the steps leading to (4.54) we obtain as $\theta \searrow 0$ and $n \to \infty$

$$\nu_{n1} \overset{\mathrm{P}}{=} o(1). \tag{4.130}$$

Again using integration by parts and (4.107)-(4.109), we can easily see that $\nu_{n2}$ of (4.129) will vanish in probability if we show that

$$\sup_{\frac{1}{n+1}\leq y<\theta} \frac{\mid \Upsilon_N(y) \mid}{q(y)} \overset{\mathrm{P}}{=} o(1), \qquad \text{as } n \to \infty, \tag{4.131}$$

for arbitrarily small $\theta$.

To verify (4.131), let $\rho_n = \frac{\log\log n}{n}$ and define the difference process $\{\Omega_n(y), 0 \leq y \leq 1\}$ by

$$\Omega_n(y) = \Upsilon_n(y) - \Gamma_n(y), \qquad 0 \leq y \leq 1, \tag{4.132}$$

where $\Gamma_.(.)$ and $\Upsilon_.(.)$ are defined by (4.6) and (4.7) respectively.

By (4.80) and (4.132) we have for any $\epsilon > 0$

$$\begin{aligned}
P & \Big( \sup_{\frac{1}{n+1}\leq y<\theta} \frac{\mid \Upsilon_N(y) \mid}{q(y)} > \epsilon\Big) \\
\leq & \; P\Big( \sup_{\frac{1}{n+1}\leq y<\theta} \frac{\mid \Upsilon_N(y) \mid}{q(y)} > \epsilon, \mid \frac{N}{n} - \tau \mid \leq a_n\Big) + b_n \\
\leq & \; P\Big( \max_{n_1\leq k\leq n_2} \sup_{\frac{1}{n+1}\leq y<\theta} \frac{\mid \Upsilon_k(y) \mid}{q(y)} > \epsilon\Big) + b_n \\
\leq & \; P\Big( \max_{n_1\leq k\leq n_2} \sup_{\frac{1}{k+1}\leq y<\theta} \frac{\mid \Upsilon_k(y) \mid}{q(y)} > \epsilon\Big) + b_n
\end{aligned}$$

$$
\begin{aligned}
&\leq P(\max_{n_1\leq k\leq n_2}\sup_{\frac{1}{k+1}\leq y<9\rho_k}\frac{\mid\Upsilon_k(y)\mid}{q(y)}>\frac{\epsilon}{2})+P(\max_{n_1\leq k\leq n_2}\sup_{9\rho_k\leq y<\theta}\frac{\mid\Upsilon_k(y)\mid}{q(y)}>\frac{\epsilon}{2})+b_n\\
&\leq P(\max_{n_1\leq k\leq n_2}\sup_{\frac{1}{k+1}\leq y<25\rho_k}\frac{\mid\Upsilon_k(y)\mid}{(\log\log k)/\sqrt{k}}.\frac{(\log\log k)/\sqrt{k}}{q(y)}>\frac{\epsilon}{2})\\
&\quad +P(\max_{k\geq n_1}\sup_{9\rho_k\leq y\leq\theta}\frac{\mid\Omega_k(y)\mid}{(y(1-y))^{\frac{1}{2}-\nu}}.\frac{(y(1-y))^{\frac{1}{2}-\nu}}{q(y)}>\frac{\epsilon}{4})\\
&\quad +P(\max_{n_1\leq k\leq n_2}\sup_{9\rho_k\leq y<\theta}\frac{\mid\Gamma_k(y)\mid}{q(y)}>\frac{\epsilon}{4})+b_n\\
&\leq P(\max_{k\geq n_1}\sup_{\frac{1}{k+1}\leq y<25\rho_k}\frac{\mid\Upsilon_k(y)\mid}{(\log\log k)/\sqrt{k}}.\sup_{0\leq y<\theta}\frac{(y^{\frac{1}{2}}\log\log\frac{1}{y})}{q(y)}>\frac{\epsilon}{2})\\
&\quad +P(\max_{k\geq n_1}\sup_{9\rho_k\leq y\leq\theta}\frac{\mid\Omega_k(y)\mid}{(y(1-y))^{\frac{1}{2}-\nu}}.\sup_{0\leq y<\theta}\frac{(y(1-y))^{\frac{1}{2}-\nu}}{q(y)}>\frac{\epsilon}{4})\\
&\quad +P(\max_{n_1\leq k\leq n_2}\sup_{9\rho_k\leq y<\theta}\sqrt{\frac{k}{n}}\frac{\mid\Gamma_k(y)\mid}{q(y)}.(\frac{1}{\sqrt{\tau-a_n}})>\frac{\epsilon}{4})+b_n, \qquad (4.133)
\end{aligned}
$$

where $n_1=n(\tau-a_n)$ and $n_2=n(\tau+a_n)$ as in (3.80) and $0<\nu<\frac{1}{2}$.

Now, using (3.10) of Csörgő and Révész (1978) and the properties of $q(.)$ for the first term in (4.133), (15.1.20) of Shorack and Wellner (1986) and the properties of $q(.)$ for the second term of (4.133) and (4.114) for the third term of (4.133) we obtain for arbitrarily small $\theta$ and any $\epsilon>0$

$$P(\sup_{\frac{1}{n+1}\leq y<\theta}\frac{\mid\Upsilon_N(y)\mid}{q(y)}>\epsilon)=o(1), \qquad \text{as } n\to\infty. \qquad (4.134)$$

Hence by (4.129) and (4.134) we get as $\theta\searrow 0$,

$$\nu_{n2}\overset{\mathrm{P}}{=}o(1), \qquad \text{as } n\to\infty. \qquad (4.135)$$

Thus by (4.126), (4.127), (4.130) and (4.135) we have for arbitrarily small $\theta$,

$$\Pi_{n1}^{(2)}\overset{\mathrm{P}}{=}o(1), \qquad \text{as } n\to\infty. \qquad (4.136)$$

We also have by the same steps leading to (4.136) and for arbitrarily small $\theta$,

$$\Pi_{n1}^{(4)} \overset{\mathrm{P}}{=} o(1), \qquad \text{as } n \to \infty, \tag{4.137}$$

where $\Pi_{n1}^{(4)}$ is defined by (4.122).

Combining (4.123)-(4.125), (4.136) and (4.137) we finish the proof of the Theorem.

Let $\mathcal{K}(.,.)$ be a Kiefer process. It is easy to see that for every $n \geq 1$ and $0 < s < 1$,

$$\int_0^1 J^*(y) \mathrm{d}\{\mathcal{K}(y, ns\tau) - s\mathcal{K}(y, n\tau)\}/(\sigma_{(J)}\sqrt{n\tau}) \overset{\mathrm{d}}{=} B(s) \tag{4.138}$$

where $B(.)$ is a standard Brownian bridge.

Hence by Theorem (4.7) and (4.138) we have

$$t_n(s) = (\frac{N}{n})^{\frac{1}{2}} T_n^*(s) \xrightarrow{\mathrm{d}} B(s), \qquad 0 < s < 1. \tag{4.139}$$

Tests against $H_1$ of (4.77) can be based on the appropriate functionals of $\{t_n(s), 0 < s < 1\}$.

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## Chapter 5

# Tests for the change point problem under ordered alternatives

### 5.1 Introduction

Let $X_1, \ldots, X_n$ be a sequence of independent rv's with unknown distribution functions (DF's), $F_1, \ldots, F_n$. The null hypothesis considered in this Chapter is given by

$$H_\circ : F_1 = \ldots = F_n = F, \textit{where } F \textit{ is unknown}.$$

Lombard (1987) and Aly and BuHamra (1996) considered the problem of testing $H_\circ$ against the unrestricted multiple change points alternative. This type of alternatives corresponds to the two-sided case in the at most one change point problem. In some situations it might be of interest to test $H_\circ$ against ordered multiple change points alternatives which are generalizations of the one-sided case in the at most one change point problem. The testing procedures of Lombard (1987) and Aly and BuHamra (1996) are change point versions of certain $k$-sample tests. Tests against ordered alternatives in the $k$-sample set-up are proposed and studied by Terpestra (1952), Jonckheere (1954), Chacko (1963) , Puri (1965) and Tryon and Hettmansperger (1973) among others. Real applications and tests of this kind of hypotheses can be found in Barlow *et al.*

(1972) and Robertson *et al.* (1988). The object of this Chapter is to propose and study testing procedures for testing against multiple change points when the changes are ordered. The alternative hypothesis considered here is given by

$$H_1 : F_1 = \ldots = F_{[nt]} \prec F_{[nt]+1} = \ldots = F_{[ns]} \prec F_{[ns]+1} = \ldots = F_n, \qquad (5.1)$$

where $[y]$ is the integer part of $y$, $\prec$ is a partial ordering on the family of DF's under consideration, $0 < t < s < 1$ and $t, s$ are unknown. This type of alternative hypotheses will be called ordered-type multiple change points alternative. We assume here that under the alternative hypothesis, the sequence has exactly two change points, but the tests we develop next can easily be extended to the case of $r > 2$ change points.

We will start first by discussing the case where the hypothesis in (5.1) involves only changes in the location parameter $\mu$ and the alternative hypothesis is given by

$$H_1 : \mu_1 = \ldots = \mu_{[nt]} < \mu_{[nt]+1} = \ldots = \mu_{[ns]} < \mu_{[ns]+1} = \ldots = \mu_n, \qquad (5.2)$$

where $0 < t < s < 1$ , $t, s$ are unknown and $\mu_1, \ldots, \mu_n$ are the location parameters of the sequence $X_1, \ldots, X_n$.

In Section 2, we formulate the testing procedure for the hypothesis in (5.2) and give some notations and assumptions. In Section 3 we derive the asymptotic theory of the test statistic proposed in Section 2. The formulation and convergence results of the test statistic for testing against $H_1$ of (5.1) are given in Section 4. In Section 5, we present the results of several Monte Carlo studies.

## 5.2 On testing $H_\circ$ against $H_1$ of (5.2)

The testing procedure proposed here is obtained by extending the test of Jonckheere (1954) and Terpestra (1952) to the change point set-up. We start by introducing the main processes needed to formulate the proposed test.

Define the two parameter processes $\{R_{in}(t,s), 0 \leq t < s \leq 1\}$, $n \geq 1$, $i = 1,2,3$ by; $R_{.n}(0,.) = R_{.n}(.,1) = 0$ and

$$\begin{aligned} R_{1n}(t,s) &= \sum_{j=1}^{[nt]} \sum_{i=[nt]+1}^{[ns]} I(X_j < X_i), & (5.3) \\ R_{2n}(t,s) &= \sum_{j=1}^{[nt]} \sum_{i=[ns]+1}^{n} I(X_j < X_i), & (5.4) \\ R_{3n}(t,s) &= \sum_{j=[nt]+1}^{[ns]} \sum_{i=[ns]+1}^{n} I(X_j < X_i), & (5.5) \end{aligned}$$

where $I(X < Y) = 1$ if $X - Y < 0$ and zero otherwise.

The proposed Jonckheere-Terpestra-type test statistic is given by

$$T_{1n} := \max_{1 \leq k < l \leq n-1} \sqrt{12} V_n(\frac{k}{n}, \frac{l}{n}), \tag{5.6}$$

where the process $\{V_n(t,s); 0 \leq t < s \leq 1\}$, is defined by, $V_n(0,.) = V_n(.,1) = 0$, and

$$V_n(t,s) = n^{-\frac{3}{2}}\{\sum_{i=1}^{3} R_{in}(t,s) - \frac{1}{2}([nt]([ns]-[nt]) + [ns](n-[ns]))\}.$$

For $0 \leq k < l \leq n$, let

$$\hat{F}_{k,l}(x) = \frac{1}{l-k} \sum_{i=k+1}^{l} I(X_i \leq x) \tag{5.7}$$

be the empirical DF based on $X_{k+1}, \cdots, X_l$ and

$$\hat{F}_{k,l}^{-1}(y) = X_{i:l-k}, \qquad \frac{i-1}{l-k} < y \le \frac{i}{l-k},\ i = 1, \ldots, l-k \tag{5.8}$$

be the corresponding quantile function, where $X_{i:m}$ is the $i^{th}$ order statistic in a sample of size $m$. In the sequel we will use the convention that

$$\int_a^b = \int_{(a,b]} .$$

By (5.3), (5.7) and (5.8) we have

$$\begin{aligned}
R_{1n}(t,s) &= \sum_{j=1}^{[nt]} \sum_{i=[nt]+1}^{[ns]} I(X_j < X_i), \\
&= \sum_{i=[nt]+1}^{[ns]} [nt] \hat{F}_{0,[nt]}(X_i) \\
&= [nt] \sum_{i=[nt]+1}^{[ns]} \hat{F}_{0,[nt]}(X_{i:[ns]-[nt]}) \\
&= [nt] \sum_{i=1}^{[ns]-[nt]} \hat{F}_{0,[nt]}(\hat{F}_{[nt],[ns]}^{-1}(y)) I\left(\frac{i-1}{[ns]-[nt]} < y \le \frac{i}{[ns]-[nt]}\right) \\
&= [nt]([ns]-[nt]) \sum_{i=1}^{[ns]-[nt]} \int_{\frac{i-1}{[ns]-[nt]}}^{\frac{i}{[ns]-[nt]}} \hat{F}_{0,[nt]}(\hat{F}_{[nt],[ns]}^{-1}(y)) \mathrm{d}y \\
&= [nt]([ns]-[nt]) \int_0^1 \hat{F}_{0,[nt]}(\hat{F}_{[nt],[ns]}^{-1}(y)) \mathrm{d}y. \qquad (5.9)
\end{aligned}$$

Similarly, for (5.4) and (5.5) we get

$$\begin{aligned}
R_{2n}(t,s) &= [nt](n-[ns]) \int_0^1 \hat{F}_{0,[nt]}(\hat{F}_{[ns],n}^{-1}(y)) \mathrm{d}y, \qquad (5.10) \\
R_{3n}(t,s) &= ([ns]-[nt])(n-[ns]) \int_0^1 \hat{F}_{[nt],[ns]}(\hat{F}_{[ns],n}^{-1}(y)) \mathrm{d}y. \qquad (5.11)
\end{aligned}$$

By (5.9), (5.10) and (5.11) we can write the process $V_n(.,.)$ of (5.6) as follows;

$$V_n(t,s) = \sum_{i=1}^{3} \int_0^1 Z_{in}(t,s,y) \mathrm{d}y, \tag{5.12}$$

where $0 < t < s < 1$, $V_n(0,.) = V_n(.,1) = 0$ and the processes $\{Z_{in}(t,s,y),\ 0 \leq t < s \leq 1,\ 0 \leq y \leq 1\}$, $i = 1,2,3$ are defined by

$$Z_{1n}(t,s,y)$$

$$= \begin{cases} \frac{[nt]([ns]-[nt])}{n^{3/2}}\{\hat{F}_{0,[nt]}(\hat{F}^{-1}_{[nt],[ns]}(y)) - y\} & ,\ 0 < y \leq 1, 0 < t < s < 1 \\ 0 & ,\ y = 0, t = 0, s = 1, \end{cases} \tag{5.13}$$

$$Z_{2n}(t,s,y)$$

$$= \begin{cases} \frac{[nt](n-[ns])}{n^{3/2}}\{\hat{F}_{0,[nt]}(\hat{F}^{-1}_{[ns],n}(y)) - y\} & ,0 < y \leq 1,\ 0 < t < s < 1 \\ 0 & ,\ y = 0, t = 0, s = 1, \end{cases} \tag{5.14}$$

and

$$Z_{3n}(t,s,y)$$

$$= \begin{cases} \frac{([ns]-[nt])(n-[ns])}{n^{3/2}}\{\hat{F}_{[nt],[ns]}(\hat{F}^{-1}_{[ns],n}(y)) - y\} & ,0 < y \leq 1,\ 0 < t < s < 1 \\ 0 & ,\ y = 0, t = 0, s = 1. \end{cases} \tag{5.15}$$

Let $F(.)$ be the common unknown DF of $H_\circ$ of (5.1). We will assume through out the rest of this Chapter that $F$ is continuous. Let $U_i = F(X_i)$, $i = 1, \ldots, n$, then under $H_\circ$ of (5.1), $U_1, \ldots, U_n$ are iid Uniform $[0,1]$ rv's. For $0 \leq k < l \leq n$, let $E_{k,l}(.)$ and $Q_{k,l}(.)$ be the uniform empirical distribution and quantile functions defined by

$$E_{k,l}(u) = \frac{1}{l-k}\sum_{i=k+1}^{l} I(U_i \leq u), \tag{5.16}$$

$$Q_{k,l}(y) = U_{i:l-k}, \qquad \frac{i-1}{l-k} < y \leq \frac{i}{l-k},\ i = 1, \ldots, l-k, \tag{5.17}$$

where $U_{i:m}$ is the $i^{th}$ order statistic in a sample of size $m$. Using (5.16) and (5.17) we define the following general uniform empirical and quantile processes $\Gamma_{k,l}(.)$ and $\Upsilon_{k,l}(.)$ by;

$$\begin{aligned}
\Gamma_{k,l}(y) &= (l-k)^{\frac{1}{2}}(E_{k,l}(y)-y), \qquad 0 \leq y \leq 1, \qquad (5.18)\\
\Upsilon_{k,l}(y) &= (l-k)^{\frac{1}{2}}(Q_{k,l}(y)-y), \qquad 0 \leq y \leq 1, \qquad (5.19)
\end{aligned}$$

where $0 \leq k < l \leq n$. Under $H_\circ$ of (5.1), we can express the processes in (5.13)-(5.15) using (5.16)-(5.19) as follows;

$$\begin{aligned}
Z_{1n}(t,s,y) &= \frac{[nt]([ns]-[nt])}{n^{\frac{3}{2}}}\{E_{0,[nt]}Q_{[nt],[ns]}(y)-y\}\\
&= \frac{([ns]-[nt])([nt])^{1/2}}{n^{3/2}}\Gamma_{0,[nt]}(Q_{[nt],[ns]}(y))\\
&- \frac{[nt]([ns])^{1/2}}{n^{3/2}}\Gamma_{0,[ns]}(y) + \frac{([nt])^{3/2}}{n^{3/2}}\Gamma_{0,[nt]}(y)\\
&+ \frac{[nt]([ns]-[nt])^{1/2}}{n^{3/2}}\{\Upsilon_{[nt],[ns]}(y) + \Gamma_{[nt],[ns]}(y)\}, \qquad (5.20)
\end{aligned}$$

$$\begin{aligned}
Z_{2n}(t,s,y) &= \frac{[nt](n-[ns])}{n^{3/2}}\{(([nt])^{-1/2}\Gamma_{0,[nt]}(Q_{[ns],n}(y))\\
&+ (n-[ns])^{-1/2}(\Upsilon_{[ns],n}(y) + \Gamma_{[ns],n}(y))\\
&- \frac{n^{1/2}}{(n-[ns])}\Gamma_{0,n}(y) + \frac{([ns])^{1/2}}{(n-[ns])}\Gamma_{0,[ns]}(y)\}, \qquad (5.21)
\end{aligned}$$

and

$$\begin{aligned} Z_{3n}(t,s,y) &= \frac{([ns]-[nt])(n-[ns])}{n^{3/2}}\{\frac{([ns])^{1/2}}{([ns]-[nt])}\Gamma_{0,[ns]}(Q_{[ns],n}(y)) \\ &+ (n-[ns])^{-1/2}(\Upsilon_{[ns],n}(y)+\Gamma_{[ns],n}(y)) \\ &- \frac{n^{1/2}}{(n-[ns])}\Gamma_{0,n}(y)+\frac{([ns])^{1/2}}{(n-[ns])}\Gamma_{0,[ns]}(y) \\ &- \frac{([nt])^{1/2}}{([ns]-[nt])}\Gamma_{0,[nt]}(Q_{[ns],n}(y))\}, \end{aligned} \tag{5.22}$$

where $0 < y \leq 1,$ and $0 < t < s < 1.$

### 5.3 The null asymptotic theory of $T_{1n}$ of (5.6)

The main result of this Section is Theorem 5.1. The proof of this Theorem depends on a number of intermediate results which will be stated and proved first.

Bahadur (1966) and Kiefer (1970), (see also Shorack and Wellner (1986)), established that as $n \to \infty$,

$$\sup_{0\leq y\leq 1} \mid \Upsilon_{0,n}(y)+\Gamma_{0,n}(y) \mid \overset{a.s.}{=} O(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}), \tag{5.23}$$

where $\Upsilon_{.,.}(.)$ and $\Gamma_{.,.}(.)$ are defined by (5.18) and (5.19), respectively.

Let, $d_{1n}(t,s) = ([ns]-[nt])$, then by (5.23) we have as $d_{1n}(t,s) \to \infty$

$$\begin{aligned} &\frac{([ns]-[nt])^{1/2}}{n^{1/2}} \sup_{0<y\leq 1} \mid \Upsilon_{[nt],[ns]}(y)+\Gamma_{[nt],[ns]}(y) \mid \\ &= \frac{(d_{1n}(t,s))^{1/2}}{n^{1/2}} \sup_{0<y\leq 1} \mid \Upsilon_{0,d_{1n}(t,s)}(y)+\Gamma_{0,d_{1n}(t,s)}(y) \mid \\ &\overset{a.s.}{=} O((d_{1n}(t,s))^{1/4}(\log d_{1n}(t,s))^{1/2}(\log\log d_{1n}(t,s))^{1/4}/n^{1/2}), \end{aligned} \tag{5.24}$$

and since (5.24) is an increasing function in the difference $d_{1n}(t,s)$, we have as $n \to \infty$

$$\begin{aligned}
& \sup_{0<t<s<1} \frac{([ns]-[nt])^{1/2}}{n^{1/2}} \sup_{0<y\leq 1} \mid \Upsilon_{[nt],[ns]}(y) + \Gamma_{[nt],[ns]}(y) \mid \\
\leq & \sup_{0<d_{1n}(t,s)<n} \frac{(d_{1n}(t,s))^{1/2}}{n^{1/2}} \sup_{0<y\leq 1} \mid \Upsilon_{0,d_{1n}(t,s)}(y) + \Gamma_{0,d_{1n}(t,s)}(y) \mid \\
& \overset{\text{a.s.}}{=} O(\sup_{0<d_{1n}(t,s)<n} (d_{1n}(t,s))^{1/4}(\log d_{1n}(t,s))^{1/2}(\log\log d_{1n}(t,s))^{1/4}/n^{1/2}) \\
\leq & \ O(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}/n^{1/2}) \\
= & \ O(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}). \qquad (5.25)
\end{aligned}$$

By Komlós, Major and Tusnády (KMT) (1975), (see also Csörgő and Révész (1981)), there exists a Kiefer process $\mathcal{K}(.,.)$ such that

$$\max_{1\leq m\leq n} \sup_{0\leq y\leq 1} \mid m^{1/2}\Gamma_{o,m}(y) - \mathcal{K}(y,m) \mid \overset{\text{a.s.}}{=} O(\log^2 n), \qquad (5.26)$$

where $\Gamma_{0,m}(.)$ is the uniform empirical process of (5.18), based on a sample of $m$ observations. Next we prove an approximation for the Kiefer processes, which we will use later. But first we should notice that, since $0 < t < s < 1$,

$$\textit{if } [nt] \to \infty \textit{ then } [ns] \to \infty,$$

$$d_{1n}(t,s) = [ns] - [nt] \geq (ns-1) - (nt+1) = n(s-t) - 2 \to \infty \ \textit{as}\, n \to \infty,$$

$$d_{2n}(s) = n - [ns] \geq n(1-s) - 1 \to \infty \ \textit{as}\, n \to \infty$$

and

$$r_n(t,s) = ([nt]/[ns]) \to (t/s) = r,\ 0 < r < 1 \ \textit{as}\, n \to \infty.$$

For the convenience of the reader, from now on we will suppress $t$ and $s$ from the notations whenever it makes no confusion. For example, we will write $d_{1n}$ for $d_{1n}(t,s) = ([ns]-[nt])$, $d_{2n}$ for $d_{2n}(s) = (n-[ns])$ and $d_{3n}$ for $d_{3n}(t) = (n-[nt])$.

**Corollary** (5.1)

For any Kiefer process $\mathcal{K}(.,.)$ we have as $n \to \infty$,

$$n^{-1/2} \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid \mathcal{K}(Q_{[nt],[ns]}(y),[nt]) - \mathcal{K}(y,[nt]) \mid \overset{\text{a.s.}}{=} O(\eta(n)),$$

where $\eta(u) = u^{-1/4}(\log u)^{1/2}(\log\log u)^{1/4}$ and $Q_{.,.}(.)$ is the uniform quantile function defined by (5.17).

**Proof**

First, we have for each $n \geq 1$, $0<t<s<1$,

$$\begin{aligned}
&\sup_{0<y\leq 1} \mid \mathcal{K}(Q_{0,d_{1n}}(y),[nt]) - \mathcal{K}(y,[nt]) \mid \\
\leq\ & \sup_{1\leq m\leq d_{1n}} \sup_{\frac{m-1}{d_{1n}}<y\leq\frac{m}{d_{1n}}} \mid \mathcal{K}(Q_{0,d_{1n}}(\frac{m}{d_{1n}}),[nt]) - \mathcal{K}(y,[nt]) \mid \\
\leq\ & \sup_{1\leq m\leq d_{1n}} \sup_{\frac{m-1}{d_{1n}}<y\leq\frac{m}{d_{1n}}} \mid \mathcal{K}(Q_{0,d_{1n}}(\frac{m}{d_{1n}}),[nt]) - \mathcal{K}(y,[nt]) \pm \mathcal{K}(\frac{m}{d_{1n}},[nt]) \mid \\
\leq\ & \sup_{1\leq m\leq d_{1n}} \mid \mathcal{K}(Q_{0,d_{1n}}(\frac{m}{d_{1n}}),[nt]) - \mathcal{K}(\frac{m}{d_{1n}},[nt]) \mid \\
&+ \sup_{1\leq m\leq d_{1n}} \sup_{\frac{m-1}{d_{1n}}<y\leq\frac{m}{d_{1n}}} \mid \mathcal{K}(\frac{m}{d_{1n}},[nt]) - \mathcal{K}(y,[nt]) \mid . \qquad (5.27)
\end{aligned}$$

By Theorem (5.1.1) of Csörgő and Révész (1981), we have, as $[nt] \to \infty$,

$$\begin{aligned}
a_{d_{1n}} &= [ns]^{1/2} \sup_{1\leq m\leq d_{1n}} \mid U_{m:d_{1n}} - \frac{m}{d_{1n}} \mid /2\{\log\log[ns]\}^{1/2} \\
&\leq \{d_{1n}^{1/2} \sup_{1\leq m\leq d_{1n}} \mid U_{m:d_{1n}} - \frac{m}{d_{1n}} \mid /2(\log\log d_{1n})^{1/2}\}\{\frac{\log\log d_{1n}}{\log\log[ns]}\frac{[ns]}{d_{1n}}\}^{1/2}
\end{aligned}$$

$$\begin{aligned} &\leq 2^{-1/2}C_1^{1/2}, \qquad \text{a.s.,} \\ &\overset{\text{a.s.}}{=} C, \end{aligned} \tag{5.28}$$

where $U_{i:k} = Q_{0,k}(\frac{i}{k})$, $i \leq k$ is defined by (5.17) and $C_1$, $C$ are positive constants. Let $\log\log(.) := \log_2(.)$, then with $b_{d_{1n}} = \sup_{1 \leq m \leq d_{1n}} \mid U_{m:d_{1n}} - \frac{m}{d_{1n}} \mid$, we have by (5.28), a.s. for all but finite number of $[nt]$,

$$\begin{aligned} &\sup_{1 \leq m \leq d_{1n}} \sup_{0 \leq s \leq b_{d_{1n}}} \mid \mathcal{K}(\frac{m}{d_{1n}} + s, [nt]) - \mathcal{K}(\frac{m}{d_{1n}}, [nt]) \mid \\ \leq \ &\sup_{1 \leq m \leq d_{1n}} \sup_{0 \leq s \leq b^*} \mid \mathcal{K}(\frac{m}{d_{1n}} + s, [nt]) - \mathcal{K}(\frac{m}{d_{1n}}, [nt]) \mid, \end{aligned} \tag{5.29}$$

where $b^* \overset{\text{a.s.}}{=} (C + o(1))((\log_2[ns])/[ns])^{1/2}$. By Theorem (1.15.2) of Csörgő and Révész (1981), we have for any Kiefer process $\mathcal{K}(.,.)$ and a sequence of positive numbers $\{h_n\}$ such that $\lim_{n\to\infty}(\log h_n^{-1})/\log_2 h_n = \infty$,

$$\begin{aligned} &\sup_{0 \leq y \leq 1-h_n} \mid \mathcal{K}(y + h_n, [nt]) - \mathcal{K}(y, [nt]) \mid / \{2[nt]h_n \log h_n^{-1}\}^{1/2} \\ \leq \ &\sup_{0 \leq y \leq 1-h_n} \sup_{0 \leq s \leq h_n} \mid \mathcal{K}(y + s, [nt]) - \mathcal{K}(y, [nt]) \mid / \{2[nt]h_n \log h_n^{-1}\}^{1/2} \\ \leq \ &1, \qquad \text{a.s.,} \quad \text{as } [nt] \to \infty. \end{aligned} \tag{5.30}$$

Taking $h_n = b^*$, we obtain from (5.28), (5.29) and (5.30), as $[nt] \to \infty$,

$$\begin{aligned} &\max_{1 \leq m \leq d_{1n}} \mid \mathcal{K}(Q_{0,d_{1n}}(\frac{m}{d_{1n}}), [nt]) - \mathcal{K}(\frac{m}{d_{1n}}, [nt]) \mid \\ \leq \ &\max_{1 \leq m \leq d_{1n}} \mid \mathcal{K}(U_{m:d_{1n}}, [nt]) - \mathcal{K}(\frac{m}{d_{1n}}, [nt]) \mid \\ \overset{\text{a.s.}}{=} \ &O([nt]^{1/2}\{(\log_2[ns])/[ns]\}^{1/4}(\log[ns])^{1/2}) \\ \leq \ &O([nt]^{1/4}\{\log_2[ns]\}^{1/4}(\log[ns])^{1/2}). \end{aligned} \tag{5.31}$$

If $\frac{m-1}{d_{1n}} < y < \frac{m}{d_{1n}}$, then $\mid y - \frac{m}{d_{1n}} \mid < \frac{1}{d_{1n}}$ and hence applying (5.30) with $h_n = d_{1n}^{-1}$ we get, as $[nt] \to \infty$,

$$\max_{1 \leq m \leq d_{1n}} \sup_{\frac{m-1}{d_{1n}} < y < \frac{m}{d_{1n}}} \mid \mathcal{K}(\frac{m}{d_{1n}}, [nt]) - \mathcal{K}(y, [nt]) \mid \overset{\text{a.s.}}{=} O(\{(\frac{[nt]}{[ns]})(\log[ns])\}^{1/2}). \quad (5.32)$$

By (5.27), (5.31) and (5.32) we get, as $[nt] \to \infty$,

$$\sup_{0 < y \leq 1} \mid \mathcal{K}(Q_{0,d_{1n}}(y), [nt]) - \mathcal{K}(y, [nt]) \mid \overset{\text{a.s.}}{=} O(\{[nt] \log_2 [ns]\}^{\frac{1}{4}} (\log[ns])^{\frac{1}{2}}). \quad (5.33)$$

Finally, since the R.H.S. of (5.33) is increasing in $[nt]$ and $[ns]$, we have as $n \to \infty$

$$\begin{aligned} & n^{-1/2} \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid \mathcal{K}(Q_{0,d_{1n}}(y), [nt]) - \mathcal{K}(y, [nt]) \mid \\ & \overset{\text{a.s.}}{=} O(n^{-1/2} \sup_{0<t<s<1} ([nt] \log_2 [ns])^{1/4} (\log[ns])^{1/2}) \\ \leq \; & O(\{n \log_2 n\}^{1/4} (\log n)^{1/2} (n^{-1/2})) \\ = \; & O(n^{-1/4} (\log_2 n)^{1/4} (\log n)^{1/2}). \end{aligned} \quad (5.34)$$

**Lemma** (5.1)

Let $\mathcal{K}(.,.)$ be the Kiefer process in (5.26), then we have as $n \to \infty$,

$$\sup_{0<t<s<1} \sup_{0<y\leq 1} \mid Z_{1n}(t, s, y) - n^{-3/2}\{[ns]\mathcal{K}(y, [nt]) - [nt]\mathcal{K}(y, [ns])\} \mid$$

$$\overset{\text{a.s.}}{=} O(n^{-1/4} (\log_2 n)^{1/4} (\log n)^{1/2}),$$

where $Z_{1n}(., ., .)$ is defined by (5.20).

**Proof**

Using the definition of $Z_{1n}(.,.,.)$ in (5.20) we have a.s.;

$$\begin{aligned}
&\sup_{0<t<s<1}\ \sup_{0<y\leq 1} \ \mid \ Z_{1n}(t,s,y) - n^{-3/2}\{[ns]\mathcal{K}(y,[nt]) - [nt]\mathcal{K}(y,[ns])\} \mid \\
\leq &\sup_{0<t<s<1}\ \sup_{0<y\leq 1} \ \Big\{ \ \frac{d_{1n}}{n^{3/2}} \mid [nt]^{1/2}\Gamma_{0,[nt]}(Q_{[nt],[ns]}(y)) - \mathcal{K}(y,[nt]) \mid \\
&\qquad + \frac{[nt]}{n^{3/2}} \mid [ns]^{1/2}\Gamma_{0,[ns]}(y) - \mathcal{K}(y,[ns]) \mid \\
&\qquad + \frac{[nt]}{n^{3/2}} \mid [nt]^{1/2}\Gamma_{0,[nt]}(y) - \mathcal{K}(y,[nt]) \mid \\
&\qquad + \frac{[nt](d_{1n})^{1/2}}{n^{3/2}} \mid \Upsilon_{[nt],[ns]}(y) + \Gamma_{[nt],[ns]}(y) \mid \Big\}. \qquad (5.35)
\end{aligned}$$

For the first term of (5.35) we have by (5.26) and Corollary (5.1), as $n \to \infty$,

$$\begin{aligned}
&\sup_{0<t<s<1}\ \sup_{0<y\leq 1} \frac{d_{1n}}{n^{3/2}} \mid [nt]^{1/2}\Gamma_{0,[nt]}(Q_{[nt],[ns]}(y)) - \mathcal{K}(y,[nt]) \mid \\
\leq \ &\sup_{0<t<s<1}\ \sup_{0<y\leq 1} n^{-1/2} \mid [nt]^{1/2}\Gamma_{0,[nt]}(Q_{[nt],[ns]}(y)) - \mathcal{K}(Q_{[nt],[ns]}(y),[nt]) \mid \\
&+ \sup_{0<t<s<1}\ \sup_{0<y\leq 1} n^{-1/2} \mid \mathcal{K}(Q_{[nt],[ns]}(y),[nt]) - \mathcal{K}(y,[nt]) \mid \\
\leq \ &\max_{1\leq m\leq n}\ \sup_{0\leq x\leq 1} n^{-1/2} \mid m^{1/2}\Gamma_{0,m}(x) - \mathcal{K}(x,m) \mid \\
&+ O_{\text{a.s.}}(n^{-1/4}(\log_2 n)^{1/4}(\log n)^{1/2}) \\
\overset{\text{a.s.}}{=} \ &O(n^{-1/2}\log^2 n) + O(n^{-1/4}(\log_2 n)^{1/4}(\log n)^{1/2}). \qquad (5.36)
\end{aligned}$$

Consequently, using (5.25), (5.26) and (5.36) in (5.35), we finish the proof.

Following the steps leading to the proof of the above lemma, we can prove the following two lemmas.

**Lemma** (5.2)

Let $\mathcal{K}(.,.)$ be the Kiefer process in (5.26), then we have as $n \to \infty$,

$$\sup_{0<t<s<1} \sup_{0<y\leq 1} \mid Z_{2n}(t,s,y) - \mathcal{K}_{2n}(t,s,y) \mid \overset{\text{a.s.}}{=} O(n^{-1/4}(\log_2 n)^{1/4}(\log n)^{1/2}),$$

where $Z_{2n}(.,.,.)$ is defined by (5.21) and,

$$\mathcal{K}_{2n}(t,s,y) = n^{-3/2}\{d_{2n}\mathcal{K}(y,[nt]) - [nt]\mathcal{K}(y,n) + [nt]\mathcal{K}(y,[ns])\}.$$

**Lemma** (5.3)

Let $\mathcal{K}(.,.)$ be the Kiefer process in (5.26), then we have as $n \to \infty$,

$$\sup_{0<t<s<1} \sup_{0<y\leq 1} \mid Z_{3n}(t,s,y) - \mathcal{K}_{3n}(t,s,y) \mid \overset{\text{a.s.}}{=} O(n^{-1/4}(\log_2 n)^{1/4}(\log n)^{1/2}),$$

where $Z_{3n}(.,.,.)$ is defined by (5.22) and,

$$\mathcal{K}_{3n}(t,s,y) = n^{-3/2}\{d_{3n}\mathcal{K}(y,[ns]) - d_{2n}\mathcal{K}(y,[nt]) - d_{1n}\mathcal{K}(y,n)\}.$$

Before we proceed to the main Theorem of this Section, we prove the following result.

**Corollary** (5.2)

Let $\mathcal{K}(.,.)$ be any Kiefer process, then as $n \to \infty$,

$$n^{-3/2} \sup_{0<t<s<1} \sup_{0\leq y\leq 1} \{ \mid \{[ns]\mathcal{K}(y,[nt]) + (n-[nt])\mathcal{K}(y,[ns]) - [ns]\mathcal{K}(y,n)\}$$
$$- \{ns\mathcal{K}(y,nt) + n(1-t)\mathcal{K}(y,ns) - ns\mathcal{K}(y,n)\} \mid \}$$
$$\overset{\text{P}}{=} O(n^{-1/2}(\log n)^{1/2}) \quad .$$

**Proof**

It is enough to prove that, as $n \to \infty$,

$$\begin{aligned} P_1(n) &= \sup_{0<t<s<1} \sup_{0 \le y \le 1} n^{-3/2} \mid (n - [nt])\mathcal{K}(y, [ns]) - (n - nt)\mathcal{K}(y, , ns) \mid \\ &\overset{\mathrm{P}}{=} O(n^{-1/2}(\log n)^{1/2}). \end{aligned} \tag{5.37}$$

For a Kiefer process $\mathcal{K}^{\circ}(.,.)$, we have by Lemma (1.11.2) of Csörgő and Révész (1981),

$$\begin{aligned} P_1(n) = \sup_{0<t<s<1} \sup_{0 \le y \le 1} n^{-1/2} & \left\{ \mid (1 - \frac{[nt]}{n})\mathcal{K}(y, [ns]) - (1-t)\mathcal{K}(y, ns) \mid \right\} \\ \overset{\mathrm{d}}{=} \sup_{0<t<s<1} \sup_{0 \le y \le 1} & \left\{ \mid (1 - \frac{[nt]}{n})\mathcal{K}^{\circ}(y, \frac{[ns]}{n}) - (1-t)\mathcal{K}^{\circ}(y, s) \mid \right\} \\ \le \sup_{0<t<s<1} \sup_{0 \le y \le 1} & \left\{ \mid (1 - \frac{[nt]}{n})\mathcal{K}^{\circ}(y, \frac{[ns]}{n}) - (1-t)\mathcal{K}^{\circ}(y, \frac{[ns]}{n}) \mid \right. \\ & \left. + \mid (1-t)(\mathcal{K}^{\circ}(y, s) - \mathcal{K}^{\circ}(y, \frac{[ns]}{n})) \mid \right\} \\ \le \sup_{0 \le x \le 1} \sup_{0 \le y \le 1} & \left\{ \mid \mathcal{K}^{\circ}(y, x) \mid \right\} \sup_{0 \le t \le 1} \mid t - \frac{[nt]}{n} \mid \\ + \sup_{0 \le t \le 1} (1-t) & \left\{ \sup_{0 \le s \le 1} \sup_{0 \le y \le 1} \mid \mathcal{K}^{\circ}(y, s) - \mathcal{K}^{\circ}(y, \frac{[ns]}{n}) \mid \right\} \\ \le O_{\mathrm{p}}(1) \,.\, \frac{1}{n} & + \sup_{0 \le y \le 1} \sup_{0 \le x \le 1-h} \sup_{0 \le h \le \frac{1}{n}} \mid \mathcal{K}^{\circ}(y, x) - \mathcal{K}^{\circ}(y, x+h) \mid \\ \overset{\mathrm{P}}{=} O(n^{-1/2}(\log n)^{1/2}). & \end{aligned}$$

**Theorem** (5.1)

Let $V_n(.,.)$ and $\mathcal{K}(.,.)$ be the processes defined in (5.12) and (5.26), respectively . Then as $n \to \infty$,

$$\begin{aligned} \sup_{0<t<s<1} & \mid V_n(t, s) - n^{-1/2} \int_0^1 \{s\mathcal{K}(y, nt) + (1-t)\mathcal{K}(y, ns) - s\mathcal{K}(y, n)\} dy \mid \\ & \overset{\mathrm{P}}{=} o(1). \end{aligned}$$

**Proof**

For $n \geq 1$ let,

$$\mathcal{K}_n^*(t,s,y) = s\mathcal{K}(y,nt) + (1-t)\mathcal{K}(y,ns) - s\mathcal{K}(y,n), \qquad 0 < t < s < 1.$$

Using the definition of $V_n(.,.)$ of (5.12), it is easy to see that,

$$\begin{aligned} \Pi_n &= \sup_{0<t<s<1} \mid V_n(t,s) - n^{-1/2}\int_0^1 \{\mathcal{K}_n^*(t,s,y)\}\mathrm{d}y \mid \\ &\leq \sup_{0<t<s<1}\sup_{0<y\leq 1} \mid \sum_{i=1}^{3} Z_{in}(t,s,y) - n^{-1/2}\{\mathcal{K}_n^*(t,s,y)\} \mid \end{aligned}$$

and hence by Lemmas (5.1)-(5.3) and Corollary (5.2) we have,

$$\begin{aligned} \Pi_n \leq\; & \sup_{0<t<s<1}\sup_{0<y\leq 1} \mid Z_{1n}(t,s,y) - n^{-3/2}\{[ns]\mathcal{K}(y,[nt]) - [nt]\mathcal{K}(y,[ns])\} \mid \\ & + \sup_{0<t<s<1}\sup_{0<y\leq 1} \mid Z_{2n}(t,s,y) - \mathcal{K}_{2n}(t,s,y) \mid \\ & + \sup_{0<t<s<1}\sup_{0<y\leq 1} \mid Z_{3n}(t,s,y) - \mathcal{K}_{3n}(t,s,y) \mid \\ & + \sup_{0<t<s<1}\sup_{0<y\leq 1} \{\mid n^{-3/2}\{[ns]\mathcal{K}(y,[nt]) - [nt]\mathcal{K}(y,[ns])\} + \mathcal{K}_{2n}(t,s,y) \\ & + \mathcal{K}_{3n}(t,s,y) - n^{-1/2}\{s\mathcal{K}(y,nt) + (1-t)\mathcal{K}(y,ns) - s\mathcal{K}(y,n)\} \mid\} \\ \overset{P}{=}\; & o(1). \end{aligned}$$

Let $\mathcal{K}(.,.)$ be a Kiefer process, $W(.)$ be a Brownian motion and $B(.)$ be a Brownian bridge. For $0 \leq s < t \leq 1$, we have

$$\begin{aligned} & \int_0^1 \{t\mathcal{K}(y,s) + (1-s)\mathcal{K}(y,t) - t\mathcal{K}(y,1)\}\mathrm{d}y \\ \overset{d}{=}\; & \frac{1}{\sqrt{12}}\{tW(s) + (1-s)W(t) - tW(1)\} \\ \overset{d}{=}\; & \frac{1}{\sqrt{12}}\{tB(s) + (1-s)B(t)\} := \frac{1}{\sqrt{12}}\mathcal{A}(s,t). \end{aligned} \tag{5.38}$$

**Corollary** (5.3)

For $0 < t < s < 1$, we have as $n \to \infty$

$$\sqrt{12}V_n(t,s) \xrightarrow{\text{d}} \mathcal{A}(t,s), \tag{5.39}$$

and

$$T_{1n} := \max_{1 \leq k < l \leq n-1} \sqrt{12}V_n(\frac{k}{n}, \frac{l}{n}) \xrightarrow{\text{d}} \sup_{0<s<t<1} \mathcal{A}(s,t), \tag{5.40}$$

where $V_n(.,.)$ and $T_{1n}$ are as in (5.6) and $\mathcal{A}(.,.)$ is as in (5.38).

The proof of this Corollary is an immediate consequence of Theorem 5.1 and (5.38).

## 5.4 A Score test for $H_1$ of (5.1)

In this section we develop a test statistic for testing $H_o$ against $H_1$ of (5.1). Recall again that we are using $d_{1n}$ for $d_{1n}(t,s) = ([ns]-[nt])$, $d_{2n}$ for $d_{2n}(s) = (n-[ns])$ and $d_{3n}$ for $d_{3n}(t) = (n-[nt])$. Assuming that the change points $[nt]$ and $[ns]$ are known, Puri (1965) suggested that we reject $H_o$ for large values of the statistic

$$\Lambda_n(t,s) = n^{-3/2}\{g_{n12}(t,s) + g_{n23}(t,s) + g_{n13}(t,s)\}, \tag{5.41}$$

where; for an arbitrary score function $J(.)$ and $\hat{F}_{.,.}(.)$ as in (5.7);

$$\begin{aligned} g_{n12}(t,s) &= [nt](d_{1n}) \int_0^1 J(y) d\{\hat{F}_{0,[nt]}(H_{t,s}^{(1)}(y)) - \hat{F}_{[nt],[ns]}(H_{t,s}^{(1)}(y))\}, \\ g_{n23}(t,s) &= (d_{1n})(d_{2n}) \int_0^1 J(y) d\{\hat{F}_{[nt],[ns]}(H_{t,s}^{(2)}(y)) - \hat{F}_{[ns],n}(H_{t,s}^{(2)}(y))\}, \\ g_{n13}(t,s) &= [nt](d_{2n}) \int_0^1 J(y) d\{\hat{F}_{0,[nt]}(H_{t,s}^{(3)}(y)) - \hat{F}_{[ns],n}(H_{t,s}^{(3)}(y))\}, \end{aligned} \tag{5.42}$$

and $H^{(1)}_{.,.}(.)$, $H^{(2)}_{.,.}(.)$ and $H^{(3)}_{.,.}(.)$ are the empirical quantile functions, defined by (5.8) based on the rv's $(X_1, \ldots, X_{[nt]}, X_{[nt]+1}, \ldots, X_{[ns]}); (X_{[nt]+1}, \ldots, X_{[ns]}, X_{[ns]+1}, \ldots, X_n)$ and $(X_1, \ldots, X_{[nt]}, X_{[ns]+1}, \ldots, X_n)$, respectively.

Since in the change point problem, $[nt]$ and $[ns]$ are not known, then a natural way to test the hypothesis in (5.1) is to develop test statistics based on the process $\{\Lambda_n(t,s), 0 \leq t < s \leq 1\}$, given by $\Lambda_n(0,.) = \Lambda_n(.,1) = 0$ and $\Lambda_n(t,s)$, of (5.41) for $0 < t < s < 1$. It is easy to see that, under $H_\circ$ the process $\Lambda_n(.,.)$ of (5.41) becomes;

$$\Lambda_n(t,s) = \int_0^1 J(y) d\{\sum_{i=1}^{3} D_{ni}(t,s,y)\}, \qquad 0 < t < s < 1, \tag{5.43}$$

where, by $E_{.,.}(.)$ of (5.16) we define,

$$\begin{aligned} D_{n1}(t,s,y) &= \frac{[nt](d_{1n})}{n^{3/2}}\{E_{0,[nt]}(Q^{(1)}_{t,s}(y)) - E_{[nt],[ns]}(Q^{(1)}_{t,s}(y))\}, && (5.44) \\ D_{n2}(t,s,y) &= \frac{(d_{1n})(d_{2n})}{n^{3/2}}\{E_{[nt],[ns]}(Q^{(2)}_{t,s}(y)) - E_{[ns],n}(Q^{(2)}_{t,s}(y))\}, && (5.45) \\ D_{n3}(t,s,y) &= \frac{[nt](d_{2n})}{n^{3/2}}\{E_{0,[nt]}(Q^{(3)}_{t,s}(y)) - E_{[ns],n}(Q^{(3)}_{t,s}(y))\}, && (5.46) \end{aligned}$$

and $Q^{(1)}_{.,.}(.), Q^{(2)}_{.,.}(.)$ and $Q^{(3)}_{.,.}(.)$ are the uniform empirical quantile functions defined by (5.17) based on the transformed rv's $\{F(X_1), \ldots, F(X_{[nt]}), F(X_{[nt]+1}), \ldots, F(X_{[ns]})\}$; $\{F(X_{[nt]+1}), \ldots, F(X_{[ns]}), F(X_{[ns]+1}), \ldots, F(X_n)\}$ and $\{F(X_1), \ldots, F(X_{[nt]}), F(X_{[ns]+1}), \ldots, F(X_n)\}$, respectively.

As in section 3 of this chapter, we can represent the processes in (5.44)-(5.46) as follows

$$D_{n1}(t,s,y) = \frac{([nt])^{\frac{1}{2}}[ns]}{n^{3/2}} \Gamma_{0,[nt]}(Q^{(1)}_{t,s}(y)) - \frac{([ns])^{\frac{1}{2}}[nt]}{n^{3/2}} \Gamma_{0,[ns]}(Q^{(1)}_{t,s}(y)), \tag{5.47}$$

$$
\begin{aligned}
D_{n2}(t,s,y) &= \frac{([ns])^{\frac{1}{2}} d_{3n}}{n^{3/2}} \Gamma_{0,[ns]}(Q^{(2)}_{t,s}(y)) - \{\frac{([nt])^{\frac{1}{2}} d_{2n}}{n^{3/2}}. \\
&\quad \Gamma_{0,[nt]}(Q^{(2)}_{t,s}(y))\} - \frac{n^{1/2} d_{1n}}{n^{3/2}} \Gamma_{0,n}(Q^{(2)}_{t,s}(y)), \qquad (5.48)
\end{aligned}
$$

$$
\begin{aligned}
D_{n3}(t,s,y) &= \frac{([nt])^{\frac{1}{2}} d_{2n}}{n^{3/2}} \Gamma_{0,[nt]}(Q^{(3)}_{t,s}(y)) + \{\frac{([ns])^{1/2}[nt]}{n^{3/2}}. \\
&\quad \Gamma_{0,[ns]}(Q^{(3)}_{t,s}(y))\} - \frac{n^{1/2}[nt]}{n^{3/2}} \Gamma_{0,n}(Q^{(3)}_{t,s}(y)), \qquad (5.49)
\end{aligned}
$$

where $\Gamma_{.,.}(.)$ is defined by (5.18) and $Q^{(.)}_{.,.}(.)$ are as in (5.44)-(5.46).

**Lemma** (5.4)

Let $\mathcal{K}(.,.)$ be the Kiefer process of (5.26), then as $n \to \infty$ we have,

$$
\begin{aligned}
l_1(n) &= \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid D_{n1}(t,s,y) - n^{-1/2}\{s\mathcal{K}(y,nt) - t\mathcal{K}(y,ns)\} \mid \\
&\overset{\mathrm{P}}{=} o(1), \qquad (5.50)
\end{aligned}
$$

$$
\begin{aligned}
l_2(n) &= \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid D_{n2}(t,s,y) - n^{-1/2}\{(1-t)\mathcal{K}(y,ns) - (1-s) \\
&\quad \mathcal{K}(y,nt) - (s-t)\mathcal{K}(y,n)\} \mid \overset{\mathrm{P}}{=} o(1), \qquad (5.51)
\end{aligned}
$$

$$
\begin{aligned}
l_3(n) &= \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid D_{n3}(t,s,y) - n^{-1/2}\{(1-s)\mathcal{K}(y,nt) + t\mathcal{K}(y,ns) \\
&\quad -t\mathcal{K}(y,n)\} \mid \overset{\mathrm{P}}{=} o(1), \qquad (5.52)
\end{aligned}
$$

where $D_{ni}(.,.,.), i = 1,2,3$ are defined by (5.47)-(5.49).

**Proof**

We explain here only the proof of (5.50), since the proofs of (5.51) and (5.52) can be done similarly.

By (5.26) and Corollaries (5.1), (5.2) we have as $n \to \infty$,

$$\begin{aligned}
l_{11}(n) &= \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid \frac{[ns]([nt])^{1/2}}{n^{3/2}} \Gamma_{0,[nt]}(Q^{(1)}_{t,s}(y)) - n^{-1/2} s\mathcal{K}(y,nt) \mid \\
&\leq \sup_{0<t<s<1} \sup_{0<y\leq 1} \{\mid \frac{[ns]([nt])^{1/2}}{n^{3/2}} \Gamma_{0,[nt]}(Q^{(1)}_{t,s}(y)) \\
&\quad - \frac{[ns]}{n^{3/2}} \mathcal{K}(Q^{(1)}_{t,s}(y),[nt]) \mid + \frac{[ns]}{n^{3/2}} \mid \mathcal{K}(Q^{(1)}_{t,s}(y),[nt]) - \mathcal{K}(y,[nt]) \mid \\
&\quad + \mid \frac{[ns]}{n^{3/2}} \mathcal{K}(y,[nt]) - n^{-1/2} s\mathcal{K}(y,nt) \mid\} \\
&\leq O_{\mathrm{a.s.}}(n^{-1/2}(\log n)^2) + O_{\mathrm{a.s.}}(n^{-1/4}(\log_2 n)^{1/4}(\log n)^{1/2}) \\
&\quad + O_{\mathrm{p}}(n^{-1/2}(\log n)^{1/2}) \overset{\mathrm{P}}{=} \mathrm{o}(1),
\end{aligned} \tag{5.53}$$

and similarly;

$$\begin{aligned}
l_{12}(n) &= \sup_{0<t<s<1} \sup_{0<y\leq 1} \mid \frac{[nt]([ns])^{1/2}}{n^{3/2}} \Gamma_{0,[ns]}(Q^{(1)}_{t,s}(y)) - n^{-1/2} t\mathcal{K}(y,ns) \mid \\
&\overset{\mathrm{P}}{=} \mathrm{o}(1).
\end{aligned} \tag{5.54}$$

By the fact that $l_1(n) \leq l_{11}(n) + l_{12}(n)$, a.s., and the results of (5.53) and (5.54) we finish the proof of (5.50).

Let the score function, $J(.)$, be bounded continuous on $(0,1]$ and admits the first derivative on $(0,1]$. Define the Gaussian process $\{G_n(t,s,y), 0 \leq t < s \leq 1, 0 \leq y \leq 1\}$ by; $G_n(0,.,.) = G_n(.,1,.) = G_n(.,.,0) = 0$ and

$$G_n(t,s,y) = n^{-1/2}\{(1-t)\mathcal{K}(y,ns) + s\mathcal{K}(y,nt) - s\mathcal{K}(y,n)\}, \tag{5.55}$$

where $0 < t < s < 1$ and $\mathcal{K}(.,.)$ is the Kiefer process. It is easy to see that

$$\begin{aligned} G_n(t,s,y) = n^{-1/2} & \{s\mathcal{K}(y,nt) - t\mathcal{K}(y,ns)\} + n^{-1/2}\{(1-t)\mathcal{K}(y,ns) \\ & - (1-s)\mathcal{K}(y,nt) - (s-t)\mathcal{K}(y,n)\} \\ & + n^{-1/2}\{(1-s)\mathcal{K}(y,nt) + t\mathcal{K}(y,ns) - t\mathcal{K}(y,n)\}, \quad (5.56) \end{aligned}$$

where the above R.H.S. is the sum of the limiting processes in (5.50)-(5.52).

**Theorem** (5.2)

Let $\mathcal{K}(.,.)$ and $\Lambda_n(.,.)$ be as in (5.26) and (5.43) respectively. Then we have as $n \to \infty$,

$$\sup_{0<t<s<1} \mid \Lambda_n(t,s) - \int_0^1 J(y)dG_n(t,s,y) \mid \overset{P}{=} o(1),$$

where $G_n(.,.,.)$ is defined by (5.56).

**Proof**

Using the definition of $\Lambda_n(.,.)$ and integration by parts, we get

$$\begin{aligned} & \sup_{0<t<s<1} \mid \Lambda_n(t,s) - \int_0^1 J(y)dG_n(t,s,y) \mid \\ \leq & \sup_{0<t<s<1} \mid \int_0^1 J(y)d\{\sum_{i=1}^{3} D_{ni}(t,s,y) - G_n(t,s,y)\} \mid \\ \leq & \sup_{0<t<s<1} \mid J(y)\{\sum_{i=1}^{3} D_{ni}(t,s,y) - G_n(t,s,y)\}|_{y=0}^{y=1} \mid \\ & + \sup_{0<t<s<1} \mid \int_0^1 \{\sum_{i=1}^{3} D_{ni}(t,s,y) - G_n(t,s,y)\}J'(y)dy \mid \\ \leq & \sup_{0<t<s<1} \sup_{0<y\leq 1} \{\mid \sum_{i=1}^{3} D_{ni}(t,s,y) - G_n(t,s,y) \mid (\mid J(y) \mid + \mid J'(y) \mid)\}. \quad (5.57) \end{aligned}$$

By the properties of the score function $J(.)$, Lemma (5.4) and (5.57) we complete the proof of the Theorem.

Let $\sigma_J^2 > 0$, be defined by

$$\sigma_J^2 = 2\int_0^1 \int_0^t s(1-t)J'(s)J'(t)\mathrm{d}s\mathrm{d}t, \qquad (5.58)$$

where $J'(.)$ is the derivative of the score function $J(.)$. Then by calculating the relevant covariance we have for $0 < t < s < 1$;

$$\frac{1}{\sigma_J}\int_0^1 J(y)\mathrm{d}G_n(t,s,y) \overset{\mathrm{d}}{=} \{sB(t) + (1-t)B(s)\} = \mathcal{A}(t,s), \qquad (5.59)$$

where $B(.)$ is a standard Brownian bridge and $\mathcal{A}(t,s)$ is as in (5.38). Now, for testing against $H_1$ of (5.1), we propose the test statistic,

$$T_{2n} = \frac{1}{\sigma_J} \max_{1\le k<l\le n-1} \Lambda_n(\frac{k}{n}, \frac{l}{n}). \qquad (5.60)$$

Hence by Theorem 5.2 and (5.59) we have,

$$T_{2n} \xrightarrow{\mathrm{d}} \sup_{0<t<s<1} \mathcal{A}(t,s). \qquad (5.61)$$

### 5.5 Monte Carlo results

The application of the Jonckheere-Terpestra-type test of (5.6) depends on the availability of the critical points of the distribution of the rv on the R.H.S. of (5.40). These critical points are not available in the literature. For this reason we conducted a Monte Carlo study to approximate the critical values of the rv on the R.H.S. of (5.40) and (5.61). We simulated 1,000 realizations of the

Brownian bridge $B(.)$ on a grid of 800 points on $[0,1]$ by generating multivariate Normal variates $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_{800})$ with covariance function, $Cov(Z_i, Z_j) = t_i(1-t_j), 0 < i < j \leq 800$, where $t_i = i/(800+1), i = 1, \ldots, 800$. For each realization we calculated $\max_{1 \leq i < j \leq 800} \{t_i Z_j + (1-t_j) Z_i\}$, ordered these values and obtained the $(1-\alpha)^{th}$ percentiles for $\alpha = 0.1, 0.05$ and $0.01$. The obtained asymptotic significance points (ASP), are shown in Table 1.

To study the applicability of the ASP calculated above we conducted another Monte Carlo study to simulate the finite sample critical values of the Jonckheere-Terpestra-type test $T_{1n}$ of (5.6). For each sample size $n = 10, 20, \ldots, 100$, we generated $5,000$ random permutations of the integers $1, 2, \ldots, n$ using the IMSL function RNPER. In each of these permutations we computed the test statistic, $T_{1n}$, then ordered the $5,000$ computed values and obtained the $(1-\alpha)^{th}$ percentiles for $\alpha = 0.1, 0.05$ and $0.01$. Table 1 contains the resulting percentiles for $n = 10, 20, \ldots, 100$. The entries of this table show that the simulated critical values of the test converge to the corresponding ASP as the sample size increases.

Table 1. Finite sample and limiting critical values for the Jonckheere-Terpestra-type test

| $n$ | $\alpha$=*0.10* | $\alpha$=*0.05* | $\alpha$=*0.01* |
|---|---|---|---|
| 10 | 1.1502 | 1.2597 | 1.5883 |
| 20 | 1.1619 | 1.3361 | 1.6460 |
| 30 | 1.1700 | 1.3281 | 1.6338 |
| 40 | 1.2118 | 1.3761 | 1.6774 |
| 50 | 1.2100 | 1.3717 | 1.6656 |
| 60 | 1.2186 | 1.3975 | 1.7105 |
| 70 | 1.2332 | 1.4106 | 1.7182 |
| 80 | 1.2175 | 1.3628 | 1.6968 |
| 90 | 1.2192 | 1.3896 | 1.7466 |
| 100 | 1.2280 | 1.3700 | 1.7580 |
| ASP | 1.2448 | 1.4571 | 1.7457 |

To measure the performance of the Jonckheere-Terpestra-type test we compared its estimated powers with the corresponding powers of six other multiple change points tests. Four of these tests were proposed by Aly and BuHamra (1996) and two were introduced by Lombard (1987). These six tests were designed to test

$$H_o : \mu_1 = \ldots = \mu_n$$

against the unrestricted version of the hypothesis of (5.2) given by

$$H_1 : \mu_1 = \ldots = \mu_{[nt]} \neq \mu_{[nt]+1} = \ldots = \mu_{[ns]} \neq \mu_{[ns]+1} = \ldots = \mu_n,$$

where $0 < t < s < 1$ are unknown. To introduce these tests, let, $r_i$ = rank of $X_i$, among $X_1, \ldots, X_n$ and define

$$R_k = \sum_{i}^{k} \frac{r_i}{n}, \qquad k = 1, \ldots, n.$$

Now consider the following processes,

$$A_n(\frac{k}{n}, \frac{l}{n}) = 12\{\frac{R_k^2}{k} + \frac{(R_l - R_k)^2}{(l-k)} + \frac{(R_n - R_l)^2}{(n-l)} - \frac{n}{4}\},$$

and

$$A_n^*(\frac{k}{n}, \frac{l}{n}) = n^{-3}k(l-k)(n-l)A_n(\frac{k}{n}, \frac{l}{n}),$$

where $1 \leq k < l \leq n-1$. Aly and BuHamra (1996)'s tests are given by

$$\begin{aligned} t_{1n} &= \max_{1 \leq k < l \leq n-1} A_n^*(\frac{k}{n}, \frac{l}{n}), \\ t_{2n} &= \frac{1}{n^2} \sum_{1 \leq k < l \leq n-1} A_n^*(\frac{k}{n}, \frac{l}{n}), \end{aligned}$$

and,

$$t_{3n}(\delta) = \frac{1}{n^5} \sum_{k=1}^{n-[n\delta]-1} \sum_{l=k+[n\delta]}^{n-1} A_n(\frac{k}{n}, \frac{l}{n}),$$

where $\delta$ was taken to be 0.1 and 0.05. Let $R_k^* = R_k - \frac{k}{2}$, $k = 1, \ldots, n$, then Lombard (1987)'s tests are given by

$$m_{2n} = \frac{12}{n^3} \sum_{1 \leq k < l \leq n-1} \{R^{*2}_{\ k} + (R_l^* - R_k^*)^2 + (R_n^* - R_l^*)^2\},$$

and,

$$m_{2n}^* = \frac{12}{n} \max_{1 \leq k < l \leq n-1} \{R^{*2}_{\ k} + (R_l^* - R_k^*)^2 + (R_n^* - R_l^*)^2\}.$$

Aly and BuHamra (1996) provided a Monte Carlo power comparison for the above mentioned six tests. We obtained Monte Carlo powers for the Jonckheere-Terpestra-type test and the above six statistics. The powers were obtained for sample size $n = 20$ from Normal, Exponential, Double-Exponential and Cauchy distributions. We considered the change points combinations $(k, l)$, $k < l = [nq]$,

$q \in \{0.05, 0.10, 0.25, 0.50, 0.75, 0.85, 0.90\}$. Six cases were considered for the size of the of the location shifts $\Delta_1$ at $k+1$ and $\Delta_2$ at $l+1$. The shift sizes were computed as the solution of the equations $P(X_{k+1} > X_1) = p_1$ and $P(X_{l+1} > X_l) = p_2$ and $p_1 \leq p_2 \in \{0.6, 0.7, 0.8\}$. We simulated the critical values of the six tests for $n = 20$, by conducting a Monte Carlo study parallel to that used to obtain the finite sample critical values of Table 1. To calculate the powers, we simulated $3,000$ realizations of samples of size $n = 20$ under the alternative distribution and computed the seven tests in each realization. Then for each test we obtained the fraction of the number of times that the null hypothesis is rejected. The results of this power study are reported in Tables 2-5.

Four general conclusions can be drawn from this Monte Carlo power study. First, in all the cases and all the four distributions, the estimated powers of the Jonckheere- Terpestra-type test $T_{1n}$ were the largest. This is not surprising since our test $T_{1n}$ is the only test designed to test for the ordered alternative-type hypothesis. Second, for any shift probabilities $(p_1, p_2)$, in all the considered distributions and test statistics, the estimated powers were largest when the second shift (or the first by symmetry) occurs in the middle $(1, 10)$, $(2, 10)$, $(5, 10)$. This is because when one of the shifts occurs in middle it should be much easier to detect the changes than if both occur near the tails of the data set. Third, as reported by Aly and BuHamra. (1996), the sup-type tests $t_{1n}$ and $m^*_{2n}$ have the lowest powers and $t_{1n}$ has higher powers than $m^*_{2n}$ except when the power reaches its maximum, they change places. Finally, the powers are higher for the Double-Exponential distribution, which may be due to its long tails, that produces larger amounts of shifts compared to the other distributions.

Table 2. Percentages of $3,000$ samples declared significant

at $n = 20$, $\alpha = 0.05$ for Normal rv's

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| (1,2) | 0.6,0.6 | 6.1 | 6.1 | 6.0 | 5.7 | 5.2 | 5.0 | 9.1 |
| | 0.7,0.7 | 9.0 | 9.0 | 8.2 | 6.0 | 6.9 | 6.7 | 15.6 |
| | 0.8,0.8 | 11.8 | 12.1 | 10.4 | 8.0 | 8.6 | 7.6 | 20.1 |
| | 0.6,0.7 | 8.2 | 8.2 | 7.7 | 6.2 | 6.9 | 7.0 | 13.7 |
| | 0.6,0.8 | 10.7 | 10.7 | 9.5 | 6.7 | 7.9 | 7.2 | 17.1 |
| | 0.7,0.8 | 10.2 | 10.4 | 9.2 | 7.2 | 7.7 | 6.9 | 17.4 |
| (1,5) | 0.6,0.6 | 8.8 | 8.8 | 8.6 | 7.2 | 7.5 | 7.7 | 14.2 |
| | 0.7,0.7 | 21.0 | 21.1 | 20.3 | 16.7 | 17.1 | 15.0 | 30.8 |
| | 0.8,0.8 | 44.4 | 44.4 | 42.3 | 35.9 | 37.1 | 28.2 | 56.6 |
| | 0.6,0.7 | 20.0 | 20.1 | 19.5 | 15.8 | 16.9 | 14.3 | 30.1 |
| | 0.6,0.8 | 38.5 | 38.6 | 38.0 | 33.0 | 33.6 | 26.7 | 50.8 |
| | 0.7,0.8 | 43.4 | 43.5 | 42.0 | 34.0 | 36.4 | 28.2 | 54.3 |
| (1,10) | 0.6,0.6 | 11.6 | 11.7 | 11.2 | 8.0 | 11.3 | 9.2 | 17.9 |
| | 0.7,0.7 | 30.2 | 30.4 | 29.7 | 19.7 | 29.6 | 23.8 | 43.5 |
| | 0.8,0.8 | 64.7 | 65.0 | 64.2 | 44.2 | 64.1 | 53.6 | 77.9 |
| | 0.6,0.7 | 28.4 | 28.7 | 28.7 | 19.8 | 29.0 | 24.1 | 41.7 |
| | 0.6,0.8 | 59.0 | 59.3 | 59.3 | 41.8 | 60.6 | 52.6 | 72.5 |
| | 0.7,0.8 | 61.4 | 61.5 | 61.7 | 43.8 | 63.2 | 53.9 | 75.4 |
| (1,15) | 0.6,0.6 | 8.9 | 9.1 | 9.0 | 7.7 | 8.4 | 6.8 | 14.6 |
| | 0.7,0.7 | 20.7 | 20.8 | 20.2 | 14.8 | 17.4 | 11.9 | 33.2 |
| | 0.8,0.8 | 43.5 | 44.0 | 42.3 | 32.4 | 37.3 | 21.7 | 60.7 |
| | 0.6,0.7 | 17.3 | 17.6 | 17.1 | 13.6 | 15.5 | 11.8 | 28.7 |
| | 0.6,0.8 | 36.7 | 37.0 | 36.3 | 30.2 | 32.1 | 23.6 | 51.4 |
| | 0.7,0.8 | 40.7 | 41.1 | 39.4 | 31.0 | 34.9 | 22.8 | 57.4 |
| (1,17) | 0.6,0.6 | 6.3 | 6.5 | 6.5 | 5.8 | 5.9 | 5.1 | 10.2 |
| | 0.7,0.7 | 11.5 | 11.7 | 11.0 | 7.4 | 9.4 | 6.1 | 20.4 |
| | 0.8,0.8 | 22.0 | 22.1 | 19.5 | 12.2 | 15.6 | 8.5 | 36.5 |
| | 0.6,0.7 | 10.2 | 10.2 | 9.6 | 7.6 | 8.3 | 6.2 | 17.3 |
| | 0.6,0.8 | 17.3 | 17.3 | 16.2 | 11.7 | 13.1 | 10.2 | 28.9 |
| | 0.7,0.8 | 18.3 | 18.6 | 16.9 | 10.8 | 13.6 | 7.9 | 31.6 |
| (1,18) | 0.6,0.6 | 5.9 | 6.1 | 5.8 | 6.1 | 5.3 | 5.0 | 10.1 |
| | 0.7,0.7 | 8.0 | 8.0 | 7.3 | 6.0 | 6.2 | 4.6 | 14.8 |
| | 0.8,0.8 | 13.0 | 13.4 | 11.3 | 5.7 | 9.0 | 4.1 | 24.2 |
| | 0.6,0.7 | 7.6 | 7.7 | 6.9 | 6.3 | 6.2 | 5.7 | 14.0 |
| | 0.6,0.8 | 10.5 | 10.5 | 9.4 | 6.6 | 8.2 | 6.2 | 19.1 |
| | 0.7,0.8 | 12.3 | 12.4 | 10.7 | 7.6 | 9.0 | 6.2 | 21.2 |

TABLE 2 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| (2, 5) | 0.6,0.6 | 10.3 | 10.3 | 10.2 | 9.0 | 9.3 | 7.9 | 17.5 |
| | 0.7,0.7 | 28.6 | 28.6 | 27.4 | 21.8 | 23.2 | 18.3 | 38.9 |
| | 0.8,0.8 | 57.3 | 57.4 | 55.1 | 45.4 | 46.1 | 37.3 | 65.6 |
| | 0.6,0.7 | 23.0 | 23.1 | 22.6 | 18.5 | 19.2 | 15.5 | 32.8 |
| | 0.6,0.8 | 41.1 | 41.0 | 40.3 | 34.8 | 35.7 | 28.4 | 54.3 |
| | 0.7,0.8 | 50.5 | 50.3 | 49.5 | 41.4 | 42.5 | 33.2 | 61.1 |
| (2, 10) | 0.6,0.6 | 13.7 | 13.7 | 13.4 | 9.2 | 13.3 | 10.6 | 22.7 |
| | 0.7,0.7 | 38.6 | 39.2 | 37.5 | 22.3 | 37.2 | 28.0 | 53.6 |
| | 0.8,0.8 | 75.4 | 75.9 | 75.4 | 52.8 | 74.7 | 60.9 | 86.2 |
| | 0.6,0.7 | 31.6 | 32.1 | 32.0 | 20.5 | 31.8 | 25.4 | 44.7 |
| | 0.6,0.8 | 61.6 | 62.0 | 62.0 | 44.0 | 62.8 | 53.4 | 76.0 |
| | 0.7,0.8 | 68.4 | 68.7 | 68.7 | 49.1 | 68.4 | 57.8 | 80.6 |
| (2, 15) | 0.6,0.6 | 9.1 | 9.2 | 8.8 | 7.2 | 8.0 | 6.8 | 16.4 |
| | 0.7,0.7 | 24.1 | 24.5 | 22.8 | 14.6 | 20.3 | 11.7 | 39.9 |
| | 0.8,0.8 | 53.9 | 54.3 | 52.2 | 35.6 | 45.6 | 25.3 | 72.7 |
| | 0.6,0.7 | 20.4 | 20.7 | 20.4 | 15.4 | 18.5 | 13.3 | 31.8 |
| | 0.6,0.8 | 39.1 | 39.5 | 38.7 | 29.3 | 33.6 | 23.0 | 55.5 |
| | 0.7,0.8 | 46.8 | 47.4 | 45.5 | 34.0 | 39.6 | 24.9 | 65.1 |
| (2, 17) | 0.6,0.6 | 7.2 | 7.4 | 7.1 | 6.7 | 6.5 | 5.3 | 13.1 |
| | 0.7,0.7 | 17.2 | 17.4 | 15.6 | 9.1 | 13.4 | 6.8 | 28.7 |
| | 0.8,0.8 | 31.4 | 31.7 | 27.9 | 14.1 | 22.2 | 10.0 | 50.1 |
| | 0.6,0.7 | 12.0 | 12.1 | 11.2 | 8.4 | 10.0 | 7.0 | 20.5 |
| | 0.6,0.8 | 20.0 | 20.1 | 18.0 | 11.4 | 14.3 | 9.3 | 32.6 |
| | 0.7,0.8 | 24.1 | 24.5 | 22.1 | 12.0 | 18.3 | 9.2 | 39.3 |
| (2,18) | 0.6,0.6 | 6.5 | 6.5 | 6.3 | 5.5 | 5.3 | 5.0 | 10.4 |
| | 0.7,0.7 | 11.6 | 11.7 | 10.9 | 6.4 | 8.7 | 5.6 | 21.3 |
| | 0.8,0.8 | 19.8 | 20.0 | 17.0 | 8.3 | 13.5 | 6.0 | 35.9 |
| | 0.6,0.7 | 8.5 | 8.7 | 8.1 | 6.5 | 7.1 | 5.2 | 15.6 |
| | 0.6,0.8 | 12.0 | 12.2 | 10.8 | 6.7 | 8.8 | 6.1 | 21.4 |
| | 0.7,0.8 | 16.2 | 16.4 | 14.9 | 7.8 | 12.0 | 6.1 | 27.9 |
| (5,10) | 0.6,0.6 | 19.4 | 19.4 | 19.4 | 12.7 | 19.4 | 15.4 | 29.8 |
| | 0.7,0.7 | 62.3 | 62.7 | 62.6 | 44.1 | 62.8 | 48.9 | 74.7 |
| | 0.8,0.8 | 95.1 | 95.2 | 95.2 | 85.7 | 95.0 | 86.4 | 97.5 |
| | 0.6,0.7 | 42.0 | 42.2 | 42.0 | 28.2 | 42.8 | 34.1 | 57.0 |
| | 0.6,0.8 | 71.1 | 71.4 | 72.2 | 54.6 | 73.3 | 63.9 | 83.1 |
| | 0.7,0.8 | 86.8 | 87.0 | 87.0 | 70.4 | 87.6 | 77.1 | 93.1 |

TABLE 2 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(5,15)$ | 0.6,0.6 | 13.8 | 14.1 | 13.5 | 8.8 | 12.6 | 8.7 | 23.6 |
| | 0.7,0.7 | 45.0 | 45.8 | 44.1 | 30.2 | 41.4 | 24.0 | 61.5 |
| | 0.8,0.8 | 81.9 | 82.2 | 80.6 | 65.4 | 77.1 | 49.5 | 91.3 |
| | 0.6,0.7 | 27.6 | 27.9 | 27.1 | 18.0 | 24.8 | 15.7 | 41.2 |
| | 0.6,0.8 | 48.7 | 49.0 | 47.2 | 35.0 | 44.0 | 28.4 | 65.0 |
| | 0.7,0.8 | 66.6 | 67.1 | 65.1 | 48.9 | 61.6 | 38.4 | 81.2 |
| $(5,17)$ | 0.6,0.6 | 10.7 | 10.8 | 10.1 | 7.6 | 9.7 | 7.2 | 18.9 |
| | 0.7,0.7 | 32.9 | 33.1 | 31.8 | 19.5 | 27.3 | 15.4 | 49.7 |
| | 0.8,0.8 | 65.1 | 65.7 | 62.3 | 40.9 | 56.6 | 28.5 | 82.7 |
| | 0.6,0.7 | 17.2 | 17.6 | 16.6 | 10.0 | 14.5 | 8.5 | 28.6 |
| | 0.6,0.8 | 26.7 | 27.3 | 24.9 | 14.8 | 21.5 | 12.6 | 42.4 |
| | 0.7,0.8 | 45.6 | 46.1 | 43.2 | 25.0 | 37.7 | 19.5 | 64.2 |
| $(5,18)$ | 0.6,0.6 | 9.1 | 9.1 | 9.0 | 7.1 | 8.2 | 6.8 | 15.3 |
| | 0.7,0.7 | 25.7 | 26.1 | 24.7 | 16.0 | 21.6 | 11.8 | 41.0 |
| | 0.8,0.8 | 53.7 | 54.0 | 51.0 | 35.5 | 45.7 | 23.0 | 72.1 |
| | 0.6,0.7 | 12.9 | 12.9 | 12.0 | 8.8 | 10.8 | 7.3 | 22.6 |
| | 0.6,0.8 | 17.3 | 17.4 | 16.1 | 9.8 | 13.7 | 8.3 | 29.4 |
| | 0.7,0.8 | 44.3 | 44.4 | 41.7 | 28.2 | 38.3 | 23.3 | 63.4 |

Table 3. Percentages of $3,000$ samples declared significant

at $n = 20$, $\alpha = 0.05$ for Exponential rv's

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | 0.6,0.6 | 6.4 | 6.4 | 5.8 | 4.5 | 5.2 | 5.0 | 8.4 |
| | 0.7,0.7 | 9.3 | 9.5 | 8.1 | 6.1 | 6.6 | 5.3 | 14.4 |
| | 0.8,0.8 | 13.6 | 13.5 | 11.3 | 7.2 | 8.8 | 7.1 | 19.8 |
| | 0.6,0.7 | 9.1 | 9.1 | 8.0 | 6.5 | 6.1 | 5.8 | 14.6 |
| | 0.6,0.8 | 12.8 | 12.7 | 11.3 | 7.6 | 9.1 | 7.4 | 18.5 |
| | 0.7,0.8 | 12.4 | 12.6 | 10.8 | 7.3 | 8.1 | 6.5 | 18.9 |
| $(1,5)$ | 0.6,0.6 | 10.7 | 11.0 | 10.1 | 8.0 | 9.1 | 7.0 | 16.5 |
| | 0.7,0.7 | 23.4 | 23.5 | 22.8 | 18.2 | 19.8 | 14.9 | 32.1 |
| | 0.8,0.8 | 45.2 | 45.1 | 43.6 | 38.0 | 38.1 | 32.0 | 55.1 |
| | 0.6,0.7 | 21.8 | 22.1 | 20.9 | 17.7 | 18.7 | 15.4 | 31.5 |
| | 0.6,0.8 | 40.1 | 40.2 | 38.6 | 34.8 | 34.5 | 28.6 | 49.8 |
| | 0.7,0.8 | 44.2 | 42.5 | 36.7 | 37.4 | 31.1 | 54.1 | 60.7 |
| $(1,10)$ | 0.6,0.6 | 10.8 | 11.0 | 10.6 | 7.8 | 10.6 | 9.3 | 18.5 |
| | 0.7,0.7 | 30.7 | 30.9 | 30.3 | 19.4 | 30.5 | 24.7 | 43.9 |
| | 0.8,0.8 | 62.4 | 62.8 | 62.0 | 44.4 | 62.8 | 53.2 | 75.2 |
| | 0.6,0.7 | 30.3 | 30.5 | 30.0 | 20.8 | 30.9 | 26.2 | 42.6 |
| | 0.6,0.8 | 57.6 | 58.0 | 58.1 | 43.5 | 59.3 | 53.2 | 71.8 |
| | 0.7,0.8 | 60.7 | 61.0 | 60.6 | 44.8 | 61.6 | 52.8 | 73.7 |
| $(1,15)$ | 0.6,0.6 | 7.3 | 7.3 | 7.2 | 6.1 | 6.4 | 5.2 | 13.0 |
| | 0.7,0.7 | 18.9 | 19.2 | 18.3 | 13.4 | 16.0 | 11.4 | 32.4 |
| | 0.8,0.8 | 40.0 | 39.9 | 38.1 | 28.5 | 32.0 | 20.5 | 58.1 |
| | 0.6,0.7 | 15.4 | 15.6 | 15.0 | 13.2 | 14.1 | 10.9 | 26.4 |
| | 0.6,0.8 | 33.5 | 34.1 | 33.2 | 27.4 | 29.1 | 22.4 | 49.1 |
| | 0.7,0.8 | 36.4 | 36.5 | 35.3 | 28.5 | 30.8 | 21.7 | 53.5 |
| (1,17) | 0.6,0.6 | 6.1 | 6.1 | 5.9 | 5.2 | 5.3 | 4.8 | 10.6 |
| | 0.7,0.7 | 11.2 | 11.2 | 10.2 | 7.4 | 8.9 | 6.4 | 19.6 |
| | 0.8,0.8 | 20.8 | 20.8 | 19.0 | 10.8 | 15.1 | 8.1 | 35.4 |
| | 0.6,0.7 | 10.0 | 10.1 | 9.5 | 7.8 | 8.0 | 7.0 | 16.9 |
| | 0.6,0.8 | 16.6 | 16.8 | 15.9 | 11.6 | 13.0 | 10.1 | 27.8 |
| | 0.7,0.8 | 18.7 | 18.8 | 17.2 | 10.6 | 13.5 | 8.2 | 30.9 |
| (1,18) | 0.6,0.6 | 6.1 | 6.2 | 6.1 | 5.5 | 5.9 | 5.5 | 8.6 |
| | 0.7,0.7 | 8.8 | 8.9 | 8.1 | 6.4 | 7.0 | 6.2 | 15.4 |
| | 0.8,0.8 | 12.5 | 12.6 | 11.2 | 6.9 | 9.1 | 4.9 | 22.9 |
| | 0.6,0.7 | 5.8 | 5.8 | 5.5 | 5.7 | 4.9 | 5.0 | 10.8 |
| | 0.6,0.8 | 9.7 | 9.9 | 8.9 | 6.6 | 7.5 | 5.6 | 17.0 |
| | 0.7,0.8 | 11.5 | 11.5 | 10.6 | 7.4 | 9.0 | 6.3 | 20.6 |

TABLE 3 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(2,5)$ | 0.6,0.6 | 12.3 | 12.4 | 11.7 | 9.0 | 10.4 | 8.4 | 18.7 |
| | 0.7,0.7 | 30.1 | 30.2 | 28.5 | 24.2 | 24.5 | 19.7 | 39.2 |
| | 0.8,0.8 | 53.4 | 53.5 | 50.9 | 45.9 | 45.1 | 37.7 | 63.8 |
| | 0.6,0.7 | 23.7 | 23.7 | 22.5 | 18.6 | 19.7 | 15.6 | 32.9 |
| | 0.6,0.8 | 42.4 | 42.5 | 40.9 | 36.6 | 36.1 | 31.0 | 53.0 |
| | 0.7,0.8 | 47.5 | 47.4 | 45.5 | 39.3 | 40.3 | 32.4 | 56.6 |
| $(2,10)$ | 0.6,0.6 | 13.0 | 13.2 | 12.4 | 8.3 | 12.2 | 9.6 | 21.2 |
| | 0.7,0.7 | 37.4 | 37.8 | 36.6 | 22.4 | 35.8 | 27.5 | 51.4 |
| | 0.8,0.8 | 70.2 | 70.5 | 69.7 | 51.9 | 69.8 | 58.7 | 83.0 |
| | 0.6,0.7 | 31.9 | 32.3 | 31.5 | 20.6 | 32.2 | 25.8 | 44.8 |
| | 0.6,0.8 | 60.8 | 61.1 | 61.2 | 44.0 | 62.6 | 55.0 | 73.9 |
| | 0.7,0.8 | 65.4 | 65.8 | 65.3 | 45.8 | 65.6 | 55.5 | 77.9 |
| $(2,15)$ | 0.6,0.6 | 9.1 | 9.2 | 8.7 | 6.9 | 7.6 | 6.6 | 16.8 |
| | 0.7,0.7 | 25.1 | 25.6 | 24.2 | 15.2 | 20.7 | 12.5 | 40.5 |
| | 0.8,0.8 | 52.5 | 53.0 | 49.7 | 33.6 | 43.4 | 24.1 | 71.3 |
| | 0.6,0.7 | 18.8 | 19.0 | 18.4 | 14.6 | 16.7 | 12.2 | 31.2 |
| | 0.6,0.8 | 38.6 | 38.8 | 37.5 | 28.3 | 32.9 | 23,0 | 54.1 |
| | 0.7,0.8 | 44.4 | 44.6 | 42.8 | 30.8 | 37.8 | 23.7 | 62.0 |
| $(2,17)$ | 0.6,0.6 | 6.8 | 7.0 | 6.5 | 5.6 | 5.2 | 4.8 | 12.2 |
| | 0.7,0.7 | 15.3 | 15.5 | 14.4 | 8.6 | 12.4 | 7.2 | 27.9 |
| | 0.8,0.8 | 31.3 | 31.6 | 27.0 | 12.6 | 20.6 | 8.4 | 52.3 |
| | 0.6,0.7 | 10.5 | 10.6 | 10.1 | 7.6 | 8.6 | 6.8 | 20.1 |
| | 0.6,0.8 | 17.4 | 17.5 | 16.0 | 9.9 | 12.9 | 8.6 | 29.9 |
| | 0.7,0.8 | 23.0 | 23.3 | 21.0 | 10.9 | 16.7 | 9.2 | 38.8 |
| (2,18) | 0.6,0.6 | 7.2 | 7.2 | 6.9 | 5.8 | 5.9 | 5.7 | 11.9 |
| | 0.7,0.7 | 13.3 | 13.7 | 12.4 | 7.3 | 10.3 | 6.2 | 22.6 |
| | 0.8,0.8 | 21.2 | 21.7 | 18.3 | 8.5 | 14.4 | 5.7 | 35.6 |
| | 0.6,0.7 | 6.9 | 7.1 | 6.7 | 5.9 | 5.5 | 5.4 | 13.7 |
| | 0.6,0.8 | 11.5 | 11.5 | 10.8 | 7.7 | 9.1 | 6.9 | 19.6 |
| | 0.7,0.8 | 15.7 | 15.9 | 14.3 | 7.6 | 11.5 | 5.8 | 29.5 |
| (5,10) | 0.6,0.6 | 18.7 | 18.9 | 18.5 | 12.2 | 18.0 | 14.5 | 28.6 |
| | 0.7,0.7 | 55.6 | 56.0 | 55.2 | 40.8 | 55.3 | 46.1 | 69.3 |
| | 0.8,0.8 | 85.7 | 85.8 | 85.8 | 73.3 | 86.3 | 77.4 | 93.3 |
| | 0.6,0.7 | 39.8 | 39.9 | 39.7 | 27.8 | 40.3 | 33.0 | 54.1 |
| | 0.6,0.8 | 66.1 | 66.3 | 66.3 | 50.6 | 67.6 | 61.2 | 78.2 |
| | 0.7,0.8 | 75.8 | 76.2 | 75.9 | 60.3 | 76.8 | 68.8 | 85.9 |

TABLE 3 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(5,15)$ | 0.6,0.6 | 13.1 | 13.4 | 12.9 | 9.5 | 12.3 | 9.7 | 22.9 |
| | 0.7,0.7 | 42.5 | 42.9 | 41.2 | 28.8 | 39.0 | 24.1 | 61.0 |
| | 0.8,0.8 | 77.2 | 77.5 | 76.1 | 60.1 | 72.6 | 47.5 | 89.1 |
| | 0.6,0.7 | 25.9 | 26.0 | 24.8 | 17.6 | 22.9 | 15.1 | 40.8 |
| | 0.6,0.8 | 45.9 | 46.4 | 44.8 | 33.4 | 41.1 | 28.3 | 61.6 |
| | 0.7,0.8 | 62.5 | 63.2 | 61.2 | 45.0 | 57.3 | 36.6 | 77.8 |
| $(5,17)$ | 0.6,0.6 | 10.8 | 11.1 | 10.4 | 7.7 | 8.9 | 6.8 | 18.5 |
| | 0.7,0.7 | 30.1 | 30.5 | 28.8 | 19.2 | 25.5 | 15.3 | 46.9 |
| | 0.8,0.8 | 63.1 | 63.4 | 61.0 | 42.7 | 55.7 | 32.2 | 78.8 |
| | 0.6,0.7 | 16.6 | 16.8 | 15.5 | 10.0 | 14.0 | 8.7 | 28.4 |
| | 0.6,0.8 | 25.1 | 25.5 | 23.6 | 14.0 | 20.8 | 12.2 | 39.3 |
| | 0.7,0.8 | 41.6 | 42.2 | 39.2 | 23.5 | 35.4 | 18.7 | 60.6 |
| $(5,18)$ | 0.6,0.6 | 9.8 | 9.9 | 9.4 | 7.4 | 8.9 | 6.3 | 17.6 |
| | 0.7,0.7 | 26.1 | 26.3 | 25.0 | 17.8 | 21.9 | 12.7 | 40.0 |
| | 0.8,0.8 | 52.1 | 52.6 | 49.1 | 35.4 | 42.8 | 26.8 | 69.7 |
| | 0.6,0.7 | 12.2 | 12.4 | 11.5 | 8.5 | 10.4 | 7.2 | 22.6 |
| | 0.6,0.8 | 16.8 | 17.0 | 15.6 | 10.0 | 13.8 | 8.5 | 28.3 |
| | 0.7,0.8 | 31.6 | 32.0 | 29.8 | 19.3 | 26.5 | 14.7 | 48.9 |

Table 4. Percentages of 3,000samples declared significant

at $n = 20$, $\alpha = 0.05$ for Double-Exponential rv's

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| (1,2) | 0.6,0.6 | 14.6 | 14.8 | 12.2 | 8.0 | 9.5 | 7.6 | 22.2 |
| | 0.7,0.7 | 14.7 | 14.5 | 12.9 | 8.1 | 9.5 | 7.5 | 22.3 |
| | 0.8,0.8 | 15.6 | 15.6 | 13.6 | 8.9 | 10.3 | 8.2 | 22.8 |
| | 0.6,0.7 | 14.9 | 14.8 | 12.6 | 9.0 | 10.1 | 8.0 | 23.4 |
| | 0.6,0.8 | 14.7 | 14.8 | 12.1 | 7.8 | 9.4 | 7.1 | 23.2 |
| | 0.7,0.8 | 15.4 | 15.5 | 13.3 | 8.5 | 10.7 | 8.2 | 24.0 |
| (1,5) | 0.6,0.6 | 69.2 | 69.1 | 66.9 | 59.8 | 59.6 | 48.8 | 76.9 |
| | 0.7,0.7 | 67.6 | 67.5 | 65.5 | 59.1 | 57.2 | 48.0 | 75.1 |
| | 0.8,0.8 | 68.5 | 68.6 | 66.0 | 57.9 | 57.2 | 47.2 | 75.9 |
| | 0.6,0.7 | 69.6 | 69.7 | 68.1 | 61.2 | 60.2 | 50.0 | 77.6 |
| | 0.6,0.8 | 71.4 | 71.4 | 69.6 | 63.3 | 60.6 | 50.7 | 79.3 |
| | 0.7,0.8 | 66.8 | 66.6 | 64.6 | 57.5 | 55.0 | 46.6 | 74.3 |
| (1,10) | 0.6,0.6 | 86.3 | 86.6 | 86.2 | 70.3 | 86.7 | 78.9 | 93.8 |
| | 0.7,0.7 | 86.7 | 86.8 | 86.7 | 69.4 | 87.2 | 78.8 | 94.1 |
| | 0.8,0.8 | 85.8 | 86.1 | 85.3 | 68.9 | 85.5 | 78.5 | 93.2 |
| | 0.6,0.7 | 87.8 | 88.0 | 87.7 | 72.8 | 88.1 | 81.5 | 94.8 |
| | 0.6,0.8 | 89.4 | 89.5 | 89.6 | 75.1 | 89.9 | 83.7 | 96.1 |
| | 0.7,0.8 | 85.0 | 85.2 | 85.1 | 68.8 | 85.7 | 77.7 | 93.1 |
| (1,15) | 0.6,0.6 | 65.6 | 65.7 | 63.2 | 52.4 | 55.8 | 35.7 | 80.9 |
| | 0.7,0.7 | 65.7 | 66.1 | 63.4 | 51.8 | 55.2 | 35.4 | 81.4 |
| | 0.8,0.8 | 64.6 | 64.9 | 62.5 | 50.7 | 55.4 | 33.8 | 81.0 |
| | 0.6,0.7 | 67.9 | 68.2 | 65.9 | 56.9 | 58.8 | 39.1 | 83.5 |
| | 0.6,0.8 | 68.7 | 69.0 | 67.1 | 56.7 | 59.7 | 39.1 | 83.6 |
| | 0.7,0.8 | 63.4 | 63.6 | 61.1 | 51.1 | 53.7 | 35.0 | 80.0 |
| (1,17) | 0.6,0.6 | 34.6 | 34.7 | 30.4 | 15.8 | 22.7 | 9.1 | 53.7 |
| | 0.7,0.7 | 34.5 | 34.8 | 30.8 | 15.0 | 22.7 | 8.6 | 53.0 |
| | 0.8,0.8 | 33.0 | 33.2 | 28.9 | 14.8 | 23.1 | 9.5 | 52.5 |
| | 0.6,0.7 | 36.0 | 36.4 | 30.9 | 15.6 | 23.5 | 9.3 | 54.9 |
| | 0.6,0.8 | 36.6 | 36.7 | 32.2 | 16.4 | 24.7 | 10.8 | 54.9 |
| | 0.7,0.8 | 33.2 | 33.4 | 28.8 | 14.3 | 22.0 | 9.7 | 51.6 |
| (1,18) | 0.6,0.6 | 18.8 | 19.0 | 16.1 | 7.8 | 12.5 | 5.6 | 32.0 |
| | 0.7,0.7 | 18.1 | 18.4 | 14.9 | 6.6 | 11.9 | 4.4 | 32.5 |
| | 0.8,0.8 | 16.8 | 17.0 | 14.0 | 7.3 | 10.9 | 4.6 | 31.6 |
| | 0.6,0.7 | 19.6 | 19.8 | 16.6 | 7.7 | 12.5 | 5.1 | 33.4 |
| | 0.6,0.8 | 19.7 | 20.0 | 16.9 | 8.1 | 13.2 | 6.4 | 34.0 |
| | 0.7,0.8 | 18.0 | 18.0 | 15.4 | 7.3 | 12.3 | 5.2 | 32.3 |

TABLE 4 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(2,5)$ | 0.6,0.6 | 79.0 | 79.1 | 76.8 | 68.9 | 67.9 | 56.6 | 84.4 |
| | 0.7,0.7 | 79.2 | 79.0 | 77.2 | 69.8 | 67.7 | 56.4 | 83.8 |
| | 0.8,0.8 | 79.7 | 79.5 | 77.0 | 68.6 | 67.0 | 56.2 | 84.2 |
| | 0.6,0.7 | 81.1 | 81.0 | 78.8 | 72.1 | 69.8 | 59.1 | 85.0 |
| | 0.6,0.8 | 81.6 | 81.4 | 79.5 | 73.5 | 70.4 | 60.1 | 86.2 |
| | 0.7,0.8 | 78.9 | 78.9 | 76.8 | 67.7 | 66.7 | 54.6 | 83.6 |
| $(2,10)$ | 0.6,0.6 | 93.2 | 93.3 | 92.5 | 76.5 | 92.0 | 83.5 | 96.9 |
| | 0.7,0.7 | 92.7 | 92.9 | 92.1 | 77.5 | 92.0 | 83.8 | 97.6 |
| | 0.8,0.8 | 92.4 | 92.7 | 91.8 | 75.2 | 91.2 | 82.1 | 96.7 |
| | 0.6,0.7 | 93.4 | 93.5 | 93.4 | 79.9 | 92.9 | 85.6 | 97.7 |
| | 0.6,0.8 | 93.8 | 94.0 | 93.3 | 80.6 | 92.8 | 86.6 | 97.4 |
| | 0.7,0.8 | 93.4 | 93.7 | 92.8 | 75.5 | 92.0 | 83.4 | 97.4 |
| $(2,15)$ | 0.6,0.6 | 77.2 | 77.7 | 74.4 | 57.1 | 67.1 | 40.6 | 90.2 |
| | 0.7,0.7 | 78.6 | 78.9 | 76.3 | 58.5 | 68.3 | 39.1 | 91.2 |
| | 0.8,0.8 | 76.4 | 76.8 | 73.4 | 55.0 | 65.5 | 37.5 | 89.8 |
| | 0.6,0.7 | 80.8 | 80.9 | 78.0 | 61.4 | 69.4 | 42.4 | 92.4 |
| | 0.6,0.8 | 80.1 | 80.4 | 77.5 | 62.8 | 70.1 | 44.8 | 92.1 |
| | 0.7,0.8 | 78.4 | 78.7 | 75.8 | 57.0 | 67.3 | 39.9 | 90.6 |
| $(2,17)$ | 0.6,0.6 | 49.6 | 50.1 | 42.8 | 18.9 | 32.4 | 11.3 | 73.3 |
| | 0.7,0.7 | 52.9 | 53.4 | 45.9 | 20.6 | 34.5 | 12.9 | 76.1 |
| | 0.8,0.8 | 49.3 | 49.6 | 43.2 | 19.2 | 32.9 | 11.7 | 74.2 |
| | 0.6,0.7 | 51.4 | 52.1 | 45.7 | 21.0 | 36.0 | 12.4 | 75.6 |
| | 0.6,0.8 | 52.1 | 52.5 | 44.8 | 18.3 | 34.6 | 11.0 | 74.7 |
| | 0.7,0.8 | 50.6 | 51.1 | 44.4 | 18.6 | 33.8 | 11.0 | 74.6 |
| (2,18) | 0.6,0.6 | 31.6 | 31.9 | 26.2 | 10.8 | 20.0 | 7.5 | 50.9 |
| | 0.7,0.7 | 32.3 | 32.8 | 26.8 | 10.8 | 19.6 | 6.6 | 52.8 |
| | 0.8,0.8 | 32.7 | 33.1 | 28.2 | 12.0 | 20.9 | 7.9 | 52.7 |
| | 0.6,0.8 | 32.7 | 33.1 | 27.5 | 10.9 | 20.1 | 7.2 | 53.0 |
| | 0.6,0.8 | 31.2 | 31.7 | 25.9 | 10.9 | 19.4 | 6.5 | 52.9 |
| | 0.7,0.8 | 32.5 | 33.0 | 27.3 | 10.3 | 19.6 | 7.0 | 51.9 |
| (5,10) | 0.6,0.6 | 98.9 | 98.9 | 99.0 | 95.4 | 99.0 | 96.4 | 99.7 |
| | 0.7,0.7 | 98.7 | 98.8 | 98.8 | 95.3 | 98.9 | 96.0 | 99.7 |
| | 0.8,0.8 | 99.2 | 99.2 | 99.2 | 96.3 | 99.2 | 96.9 | 99.8 |
| | 0.6,0.7 | 99.3 | 99.4 | 99.3 | 95.8 | 99.2 | 97.1 | 99.8 |
| | 0.6,0.8 | 99.7 | 99.7 | 99.7 | 96.9 | 99.7 | 97.3 | 100.0 |
| | 0.7,0.8 | 99.2 | 99.3 | 99.3 | 95.7 | 99.3 | 96.2 | 99.8 |

TABLE 4 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(5,15)$ | 0.6,0.6 | 95.4 | 95.6 | 95.1 | 89.2 | 94.0 | 74.9 | 98.9 |
| | 0.7,0.7 | 97.5 | 97.6 | 97.0 | 91.4 | 95.9 | 75.9 | 99.4 |
| | 0.8,0.8 | 96.5 | 96.7 | 96.0 | 91.1 | 95.3 | 77.3 | 99.2 |
| | 0.6,0.7 | 95.9 | 96.0 | 95.5 | 88.6 | 94.1 | 74.3 | 99.1 |
| | 0.6,0.8 | 96.2 | 96.3 | 95.7 | 88.8 | 94.2 | 74.5 | 98.9 |
| | 0.7,0.8 | 96.1 | 96.2 | 95.6 | 89.3 | 94.4 | 74.0 | 98.9 |
| $(5,17)$ | 0.6,0.6 | 86.7 | 86.9 | 85.1 | 66.6 | 79.0 | 47.4 | 95.2 |
| | 0.7,0.7 | 90.5 | 90.9 | 88.8 | 73.4 | 83.6 | 54.7 | 96.8 |
| | 0.8,0.8 | 89.9 | 90.2 | 88.4 | 72.2 | 83.1 | 52.6 | 96.6 |
| | 0.6,0.7 | 88.0 | 88.1 | 85.7 | 67.1 | 79.2 | 48.7 | 96.1 |
| | 0.6,0.8 | 88.1 | 88.4 | 85.8 | 66.4 | 79.5 | 46.6 | 96.2 |
| | 0.7,0.8 | 89.6 | 89.8 | 87.3 | 70.7 | 81.4 | 51.6 | 96.4 |
| $(5,18)$ | 0.6,0.6 | 78.0 | 78.5 | 75.1 | 57.6 | 67.2 | 38.5 | 91.0 |
| | 0.7,0.7 | 81.9 | 82.2 | 79 .4 | 64.0 | 72.0 | 44.3 | 92.7 |
| | 0.8,0.8 | 81.6 | 81.7 | 79.1 | 64.6 | 71.3 | 45.0 | 92.4 |
| | 0.6,0.7 | 77.8 | 78.0 | 75.2 | 57.5 | 67.3 | 40.0 | 90.6 |
| | 0.6,0.8 | 79.8 | 80.5 | 77.3 | 60.2 | 68.8 | 41.5 | 92.1 |
| | 0.7,0.8 | 80.7 | 81.1 | 78.5 | 63.3 | 70.8 | 44.6 | 92.1 |

Table 5:Percentages of 3,000 samples declared significant

at $n = 20$, $\alpha = 0.05$ for Cauchy rv's

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(1,2)$ | 0.6,0.6 | 6.1 | 6.4 | 5.9 | 5.3 | 5.8 | 5.5 | 9.6 |
| | 0.7,0.7 | 8.9 | 8.9 | 8.5 | 6.5 | 6.7 | 6.5 | 13.6 |
| | 0.8,0.8 | 12.2 | 12.1 | 10.5 | 7.2 | 8.4 | 7.1 | 18.3 |
| | 0.6,0.7 | 6.6 | 6.6 | 6.2 | 5.5 | 5.2 | 5.9 | 11.0 |
| | 0.6,0.8 | 11.1 | 11.2 | 9.6 | 7.5 | 8.2 | 7.1 | 17.8 |
| | 0.7,0.8 | 10.6 | 10.8 | 9.4 | 6.8 | 7.5 | 6.0 | 18.8 |
| $(1,5)$ | 0.6,0.6 | 8.3 | 8.3 | 8.5 | 8.0 | 7.8 | 7.4 | 14.3 |
| | 0.7,0.7 | 22.1 | 22.3 | 21.8 | 18.1 | 19.1 | 16.3 | 31.6 |
| | 0.8,0.8 | 52.3 | 52.0 | 50.7 | 45.0 | 43.8 | 37.9 | 62.7 |
| | 0.6,0.7 | 18.2 | 18.4 | 16.1 | 16.2 | 14.2 | 28.2 | 54.3 |
| | 0.6,0.8 | 45.2 | 45.1 | 43.8 | 39.8 | 38.7 | 33.2 | 57.3 |
| | 0.7,0.8 | 47.3 | 47.2 | 46.6 | 41.2 | 40.7 | 33.8 | 58.2 |
| $(1,10)$ | 0.6,0.6 | 10.5 | 10.7 | 10.7 | 7.9 | 10.7 | 9.8 | 18.3 |
| | 0.7,0.7 | 30.0 | 30.4 | 30.0 | 20.5 | 30.3 | 25.3 | 43.9 |
| | 0.8,0.8 | 69.0 | 69.2 | 68.8 | 52.3 | 69.7 | 62.0 | 81.6 |
| | 0.6,0.7 | 27.3 | 27.6 | 27.8 | 18.2 | 28.3 | 23.6 | 40.1 |
| | 0.6,0.8 | 65.1 | 65.3 | 65.5 | 50.7 | 67.1 | 62.8 | 78.5 |
| | 0.7,0.8 | 66.9 | 67.1 | 67.2 | 51.4 | 68.4 | 62.4 | 78.1 |
| $(1,15)$ | 0.6,0.6 | 8.0 | 8.2 | 8.0 | 7.6 | 7.4 | 6.9 | 14.6 |
| | 0.7,0.7 | 21.2 | 21.4 | 20.6 | 15.2 | 18.1 | 12.1 | 33.4 |
| | 0.8,0.8 | 52.7 | 53.0 | 51.2 | 42.1 | 45.1 | 30.9 | 67.8 |
| | 0.6,0.7 | 19.0 | 19.1 | 18.5 | 14.3 | 17.2 | 12.4 | 30.1 |
| | 0.6,0.8 | 44.7 | 44.9 | 43.7 | 40.2 | 37.9 | 31.3 | 58.3 |
| | 0.7,0.8 | 47.9 | 48.0 | 46.9 | 39.5 | 42.0 | 29.8 | 62.7 |
| (1,17) | 0.6,0.6 | 6.2 | 6.4 | 6.3 | 5.8 | 5.7 | 5.2 | 10.2 |
| | 0.7,0.7 | 11.5 | 11.7 | 10.9 | 7.4 | 9.3 | 5.8 | 21.0 |
| | 0.8,0.8 | 27.4 | 27.6 | 24.4 | 13.1 | 18.9 | 8.9 | 42.5 |
| | 0.6,0.7 | 10.4 | 10.5 | 10.0 | 8.0 | 8.3 | 7.0 | 17.9 |
| | 0.6,0.8 | 22.5 | 22.8 | 20.9 | 12.8 | 16.9 | 10.4 | 35.7 |
| | 0.7,0.8 | 23.6 | 23.9 | 21.2 | 13.8 | 17.3 | 10.4 | 36.5 |
| (1,18) | 0.6,0.6 | 5.0 | 5.1 | 4.8 | 4.7 | 4.7 | 4.4 | 8.8 |
| | 0.7,0.7 | 8.1 | 8.4 | 7.7 | 5.4 | 6.2 | 4.8 | 15.0 |
| | 0.8,0.8 | 16.9 | 17.0 | 14.8 | 7.3 | 11.5 | 5.2 | 29.0 |
| | 0.6,0.7 | 7.9 | 7.9 | 7.1 | 5.6 | 5.9 | 5.3 | 13.1 |
| | 0.6,0.7 | 12.1 | 12.3 | 10.9 | 7.3 | 8.9 | 6.8 | 20.9 |
| | 0.7,0.8 | 12.9 | 12.9 | 11.2 | 6.9 | 9.5 | 5.7 | 23.3 |

TABLE 5 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(2,5)$ | 0.6,0.6 | 10.0 | 10.1 | 10.0 | 8.6 | 8.7 | 7.5 | 16.6 |
| | 0.7,0.7 | 25.3 | 25.5 | 24.4 | 20.1 | 20.7 | 17.4 | 34.7 |
| | 0.8,0.8 | 58.4 | 58.5 | 56.3 | 49.3 | 49.1 | 41.7 | 66.8 |
| | 0.6,0.7 | 21.0 | 21.1 | 20.8 | 17.6 | 18.2 | 15.9 | 31.5 |
| | 0.6,0.8 | 47.5 | 47.8 | 47.1 | 43.3 | 41.5 | 35.4 | 58.0 |
| | 0.7,0.8 | 52.4 | 52.6 | 51.2 | 45.8 | 44.8 | 38.6 | 62.6 |
| $(2,10)$ | 0.6,0.6 | 14.4 | 14.5 | 14.1 | 9.9 | 13.8 | 11.7 | 23.3 |
| | 0.7,0.7 | 36.6 | 36.8 | 36.0 | 21.9 | 35.6 | 27.5 | 49.6 |
| | 0.8,0.8 | 76.2 | 76.6 | 76.1 | 56.8 | 75.5 | 66.0 | 86.7 |
| | 0.6,0.7 | 30.3 | 30.4 | 30.6 | 21.1 | 31.2 | 26.4 | 43.4 |
| | 0.6,0.8 | 69.1 | 69.4 | 69.5 | 53.7 | 70.9 | 64.8 | 81.8 |
| | 0.7,0.8 | 70.4 | 70.5 | 70.5 | 54.8 | 71.3 | 63.6 | 82.1 |
| $(2,15)$ | 0.6,0.6 | 8.9 | 8.9 | 8.9 | 7.8 | 8.4 | 7.6 | 16.5 |
| | 0.7,0.7 | 26.1 | 26.7 | 24.9 | 15.9 | 22.0 | 13.4 | 42.1 |
| | 0.8,0.8 | 61.3 | 61.6 | 58.5 | 45.3 | 52.7 | 33.0 | 77.8 |
| | 0.6,0.7 | 20.9 | 21.2 | 20.7 | 15.1 | 18.7 | 13.2 | 32.7 |
| | 0.6,0.8 | 48.4 | 48.5 | 48.0 | 40.4 | 42.0 | 32.9 | 62.8 |
| | 0.7,0.8 | 51.6 | 52.1 | 50.3 | 41.0 | 44.1 | 31.0 | 67.4 |
| $(2,17)$ | 0.6,0.6 | 7.4 | 7.5 | 7.0 | 5.8 | 6.3 | 5.3 | 11.9 |
| | 0.7,0.7 | 16.4 | 16.5 | 15.0 | 8.7 | 12.1 | 6.4 | 27.6 |
| | 0.8,0.8 | 38.9 | 39.2 | 34.2 | 14.5 | 25.4 | 9.8 | 59.9 |
| | 0.6,0.7 | 11.5 | 11.7 | 10.8 | 7.6 | 9.3 | 6.7 | 20.3 |
| | 0.6,0.8 | 25.4 | 25.6 | 23.0 | 14.0 | 18.1 | 11.7 | 38.6 |
| | 0.7,0.8 | 29.7 | 30.0 | 26.7 | 14.1 | 21.2 | 10.3 | 46.0 |
| (2,18) | 0.6,0.6 | 6.6 | 6.7 | 6.5 | 5.4 | 5.8 | 5.4 | 11.0 |
| | 0.7,0.7 | 12.0 | 12.3 | 10.6 | 6.9 | 8.6 | 5.7 | 22.1 |
| | 0.8,0.8 | 25.8 | 26.2 | 21.7 | 10.0 | 16.9 | 6.7 | 42.8 |
| | 0.6,0.7 | 8.4 | 8.7 | 7.9 | 6.2 | 7.2 | 5.4 | 14.5 |
| | 0.6,0.8 | 13.2 | 13.3 | 11.5 | 7.4 | 9.6 | 6.2 | 23.8 |
| | 0.7,0.8 | 17.0 | 17.3 | 15.1 | 8.8 | 11.6 | 6.0 | 30.7 |
| (5,10) | 0.6,0.6 | 18.6 | 18.8 | 18.8 | 12.4 | 18.6 | 15.0 | 28.7 |
| | 0.7,0.7 | 53.4 | 53.9 | 53.1 | 38.4 | 53.0 | 42.8 | 65.8 |
| | 0.8,0.8 | 86.9 | 87.1 | 86.8 | 73.7 | 87.2 | 77.5 | 93.2 |
| | 0.6,0.7 | 39.1 | 39.4 | 39.4 | 26.5 | 40.0 | 32.9 | 53.0 |
| | 0.6,0.8 | 72.1 | 72.5 | 72.6 | 58.5 | 74.6 | 67.2 | 83.2 |
| | 0.7,0.8 | 78.2 | 78.4 | 78.1 | 62.9 | 78.9 | 71.7 | 87.2 |

TABLE 5 (continued)

| $(k,l)$ | $(p_1,p_2)$ | $t_{3n}(0.1)$ | $t_{3n}(0.05)$ | $t_{2n}$ | $t_{1n}$ | $m_{2n}$ | $m^*_{2n}$ | $T_{1n}$ |
|---|---|---|---|---|---|---|---|---|
| $(5,15)$ | 0.6,0.6 | 14.8 | 15.0 | 14.5 | 10.2 | 13.6 | 9.4 | 24.2 |
| | 0.7,0.7 | 41.5 | 42.1 | 41.0 | 29.1 | 38.1 | 24.3 | 58.0 |
| | 0.8,0.8 | 80.8 | 81.2 | 79.2 | 65.5 | 76.7 | 54.5 | 90.8 |
| | 0.6,0.7 | 27.2 | 27.5 | 26.5 | 18.8 | 24.4 | 16.3 | 39.8 |
| | 0.6,0.8 | 54.5 | 54.6 | 53.7 | 45.0 | 50.0 | 36.7 | 68.1 |
| | 0.7,0.8 | 64.3 | 64.7 | 63.2 | 50.0 | 59.0 | 41.1 | 77.6 |
| $(5,17)$ | 0.6,0.6 | 10.1 | 10.0 | 9.7 | 7.1 | 9.2 | 6.9 | 18.3 |
| | 0.7,0.7 | 30.6 | 31.1 | 29.4 | 18.3 | 26.3 | 15.5 | 46.5 |
| | 0.8,0.8 | 68.9 | 69.6 | 66.2 | 47.4 | 61.0 | 34.9 | 84.3 |
| | 0.6,0.7 | 16.6 | 16.8 | 15.6 | 10.1 | 13.8 | 9.0 | 28.1 |
| | 0.6,0.8 | 31.8 | 32.0 | 29.4 | 16.5 | 25.3 | 13.4 | 46.4 |
| | 0.7,0.8 | 46.0 | 46.4 | 43.6 | 24.3 | 37.7 | 18.8 | 62.6 |
| $(5,18)$ | 0.6,0.6 | 9.7 | 9.8 | 9.5 | 7.4 | 8.2 | 7.0 | 16.5 |
| | 0.7,0.7 | 25.6 | 25.9 | 24.3 | 16.6 | 21.3 | 14.0 | 40.6 |
| | 0.8,0.8 | 60.9 | 61.5 | 58.6 | 42.9 | 52.1 | 30.6 | 77.3 |
| | 0.6,0.7 | 13.4 | 13.7 | 12.9 | 8.1 | 11.3 | 7.3 | 23.6 |
| | 0.6,0.8 | 18.3 | 18.5 | 17.1 | 9.6 | 15.2 | 7.9 | 31.8 |
| | 0.7,0.8 | 33.0 | 33.4 | 30.9 | 16.5 | 25.1 | 13.4 | 49.9 |

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# Chapter 6

# General discussion and topics for further research

As we read throughout the main chapters of this thesis, we can see that the change point problem arises in many statistical analysis fields. In this thesis we only discussed the detection of a possible change in either, the parameters of linear models or in the distribution function. Next, we discuss generally the work done in each part of the thesis and also point out some of the related work, that can be done in the future.

In Chapters 2 and 3, we proposed and studied test statistics which may be used to detect shifts in the parameters of the linear regression model. First in Chapter 2, we introduced weighted nonparametric tests which are sensitive to changes in the slope that occur close to the end of the data set. We also proposed test statistics to test against the epidemic-type change and the at most two change points in the slope of a simple regression line. We can see that all the tests discussed in Chapter 2, deal only with the possible slope change in simple regression models. This raises the natural question; Can we extend these statistics to the general linear model parameters ?. The answer of this question may start by extending the convergence results obtained in Chapter 2. Testing a change in the parameters of nonlinear models may be considered as another extension of these proposed tests. We may also be concerned of how the usual regression assumptions (if violated) will affect the change point tests. Lombard and Hart (1994), studied the change point problem in simple location model, when the error series are (dependent) weakly stationary. They discussed the least squares estimator of the change point and the detection of the change point. In addition, they proposed a test for the uncorrelatedness of the errors. In their application, they show that ignoring the dependence in the data may invert the decision. Tang and MacNeill (1993), studied the effect of serial correlation on tests for parameters change in linear models. They show that failure to account for the effects of serial correlation among the data will invalidate the change point tests. Thus it is important to check for the validity of the regression assumptions and account for any violation before we proceed to the change point analysis.

In Chapter 3, we obtained a Bayesian likelihood ratio (BLR) test to detect an epidemic-type change in the parameters of the general linear model. The asymptotic distribution of the BLR test was obtained when the involved weight function is assumed to be continuous. Since this does not cover all the possible weight functions, more investigation for the asymptotic distribution is required. We also note that the theoretical asymptotic quantiles were only tractable in certain special cases. A few numerical methods have been developed to calculate these asymptotic quantiles, but they are still difficult to apply. MacNeill (1974), numerically calculated selected percentage points for some BLR tests.

His method depends on solving a certain differential equation then inverting the characteristic function numerically. Jandhyala and Minogue (1993) proposed a numerical procedure to compute the quantiles of the asymptotic distributions of the BLR tests in case of general polynomial regression. In our Monte Carlo study we used a simple procedure to approximate the BLR test quantiles.

In Chapter 4, we proposed linear rank statistics for the two-sample problem when the sample sizes are random. We also proposed linear rank statistics for the at most one change point when the sample size is random. As a generalization we may study linear rank statistics in case of multiple change points and ordered-type alternative when the sample size is random.

In Chapter 5, we proposed new test statistics for ordered multiple-change point problems. We first developed tests for a multiple-change in the location parameter of a random sequence, based on Jonckheere-Terpstra test. We also obtained tests for the general multiple-change in distribution functions. The latter test is an extension of Puri (1965) $k$-sample test to the change point problem. To examine the performance of these proposed tests, we conducted a Monte Carlo study. This study supported our objective; that the proposed tests which are designed specially for the ordered-alternative change point problem are superior to the unrestricted counterparts. A similar problem which is not discussed in this thesis, is testing against Umbrella ordering. In the literature, this up-then-down response pattern has many real applications. For example, evaluating marginal gain in efficiency as a function of training degree, treatment effectiveness as a function of time and reaction to increasing age on the performance of a certain task. Following the idea of Chapter 5, we may develop tests for the Umbrella ordering change points based on the existing $k$-sample tests for the Umbrella alternatives which are available in the literature. Because of the importance of this type of inferences in applications, more investigations are required in this area.

Finally, applying the tests developed in this thesis on real data sets and examining its performance is required. We can also see that, although it is already difficult to derive the asymptotic distributions of the proposed tests under the null hypothesis, the question remains; what is the asymptotic distributions under (a set of local) alternatives ?.

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